

THE BLOCH-OKOUNKOV CORRELATION FUNCTIONS OF NEGATIVE LEVELS

SHUN-JEN CHENG, DAVID G. TAYLOR, AND WEIQIANG WANG

ABSTRACT. Bloch and Okounkov introduced an n -point correlation function on the fermionic Fock space and found a closed formula in terms of theta functions. This function affords several distinguished interpretations and in particular can be formulated as correlation functions on irreducible $\widehat{\mathfrak{gl}}_\infty$ -modules of level one. These correlation functions have been generalized for irreducible integrable modules of $\widehat{\mathfrak{gl}}_\infty$ and its classical Lie subalgebras of positive levels by the authors. In this paper we extend further these results and compute the correlation functions as well as the q -dimensions for modules of $\widehat{\mathfrak{gl}}_\infty$ and its classical subalgebras at negative levels.

1. INTRODUCTION

1.1. Bloch-Okounkov [BO] (also [Ok]) formulated an n -point correlation function on the fermionic Fock space and found a beautiful closed formula for it in terms of Jacobi theta functions. This function has since made appearances in several distinct setups including Gromov-Witten theory and Hilbert schemes. The viewpoint taken in that paper is to regard the original Bloch-Okounkov functions as correlation functions on irreducible $\widehat{\mathfrak{gl}}_\infty$ -modules of level one. In this sense, these correlation functions have been generalized for irreducible integrable modules of $\widehat{\mathfrak{gl}}_\infty$ at positive levels [CW], and also generalized for integrable modules of classical Lie subalgebras of $\widehat{\mathfrak{gl}}_\infty$ (of type B, C, D) at positive levels [TW, W2] (also cf. [Mil]). These Bloch-Okounkov functions can be also viewed as a refined version of character formulas for the corresponding modules.

The Lie algebra $\widehat{\mathfrak{gl}}_\infty$ and its classical subalgebras were introduced by the Kyoto school [DJKM1, DJKM2] in connection with vertex operators and KP integrable hierarchies, etc., and they have played fundamental roles in representation theory of infinite-dimensional Lie algebras. They are also intimately related to the $W_{1+\infty}$ algebra and its classical subalgebras (cf. [KWY] and references therein).

1.2. The goal of this paper is to study the Bloch-Okounkov correlation functions for (mostly) irreducible highest weight modules of negative levels. There are several dualities [W1] (also see [KR] for type A) on various Fock spaces of bosonic ghosts (also called $\beta\gamma$ systems in physics literature) between a finite-dimensional Lie group on one side and $\widehat{\mathfrak{gl}}_\infty$, or one of its classical subalgebras, on the other side.

These dualities are natural infinite-dimensional generalizations of the classical Howe duality for finite-dimensional Lie groups/algebras [Ho1, Ho2]. The modules of $\widehat{\mathfrak{gl}}_\infty$ and other Lie algebras under consideration in the present paper appear in these Howe duality decompositions. In fact, these dualities are the essential tools that allow us to reduce the calculation of the correlation functions of a general negative level to those of the bottom levels (i.e. level -1 or $-\frac{1}{2}$), and the precise relation involves summation over the Weyl group of the corresponding Lie group in Howe duality. Similar ideas have been used in [CW, TW].

We show that the n -point correlation functions at the bottom levels satisfy certain q -difference equations (such an idea goes back to [BO, Ok] in the original setup). We are able to compute the 1-point and 2-point correlation functions at the bottom levels explicitly in terms of certain q -hypergeometric series (cf. [GR]), but the general n -point case remains open. The negative level case is technically more complicated than the positive level case treated in our earlier works [CW, TW], and the difference between negative level and positive level is already apparent at their bottom levels.

The modules considered in this paper all possess a natural \mathbb{Z}_+ -grading with finite-dimensional subspaces, and thus it makes sense to ask for their q -dimension (i.e. graded dimension). The strategy used to calculate the correlation functions allows us to determine explicitly the q -dimensions (which can be regarded as the 0-point correlation function) for the corresponding modules. The q -dimension formula at the bottom level has been folklore, but the general case appears to be new. Indeed, the q -dimension formula at the bottom level boils down to an intriguing q -series identity, which affords several different proofs to date [BCMN, FeF, K1]. Each of these proofs is complicated yet very interesting in its own way, using *super* boson-fermion correspondence or the underlying Virasoro algebra structure of the Fock space of bosonic ghosts, just as the celebrated Jacobi triple product identity underlies the boson-fermion correspondence. Here we offer a very short combinatorial proof of this remarkable identity.

The consideration of the c_∞ -modules of level $l - \frac{1}{2}$ in this paper, which is the only positive level case left out of [TW] since it involves bosonic Fock space, also helps to complete the study of the correlation functions and q -dimensions for integrable modules of all classical Lie subalgebras of $\widehat{\mathfrak{gl}}_\infty$.

1.3. This paper is organized as follows. We treat the n -point correlation functions for $\widehat{\mathfrak{gl}}_\infty$ -modules of negative level in Section 2. The same is then done in Section 3 for the classical Lie subalgebra of $\widehat{\mathfrak{gl}}_\infty$ of type C and in Section 4 for type D . We note that the subalgebra of type B of negative level does not feature in a Howe duality and thus does not appear in this work. Along the way, q -dimension formulas are also given in each case.

1.4. Notations. For a classical simple Lie algebra, we use the standard notation to denote the roots by $\varepsilon_i - \varepsilon_j$, $\pm\varepsilon_i$ and $\pm 2\varepsilon_i$ etc. By (\cdot, \cdot) we mean the usual symmetric bilinear form determined by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$ and write $\|x\|^2 = (x, x)$. Further we let $\rho = \sum_{i=1}^l (l-i)\varepsilon_i$, $\rho_B = \sum_{i=1}^l (l-i+\frac{1}{2})\varepsilon_i$, and $\rho_C = \sum_{i=1}^l (l-i+1)\varepsilon_i$.

Given a Lie algebra x_∞ with $x = a, c, d$, we denote $L(x_\infty; \Lambda, k)$ the irreducible x_∞ -module of highest weight Λ and level k .

We denote by \mathbb{N} the set of natural numbers and by \mathbb{Z}_+ the set of nonnegative integers.

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2. THE a_∞ -CORRELATION FUNCTIONS AND q -DIMENSION FORMULAS

2.1. Some q -series identities. We start with some combinatorial preparation. For an indeterminate q we let

$$\begin{aligned} (a)_0 &= 1, \\ (a)_n &= (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad \text{for } n \in \mathbb{N}, \\ (a)_\infty &= (1-a)(1-aq)(1-aq^2)\cdots = \prod_{i=0}^{\infty} (1-aq^i). \end{aligned}$$

Alternatively we may also regard q as a complex number with $|q| < 1$ to ensure the functions in this paper converge as analytic functions. For $r, s \in \mathbb{Z}_+$, recall the q -hypergeometric series (cf. [GR])

$${}_r\Phi_s(a_1, \dots, a_r; b_1, \dots, b_s; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n (q)_n} \left((-1)^n q^{n(n-1)/2} \right)^{1+s-r} z^n.$$

In the sequel we will frequently use the following well-known identities (cf. [GR])

$$(2.1) \quad \sum_{m=0}^{\infty} \frac{(-z)^m q^{m(m-1)/2}}{(q)_m} = (z)_\infty, \quad \sum_{l=0}^{\infty} \frac{(a)_l}{(q)_l} z^l = \frac{(az)_\infty}{(z)_\infty}.$$

Proposition 2.1. *For a given $k \geq 0$, we have*

$$(2.2) \quad \sum_{l=0}^{\infty} \frac{q^l}{(q)_l (tq)_{l+k}} = \frac{1}{(q)_\infty (tq)_\infty} \sum_{m \geq 0} (-1)^m q^{m(m+1)/2 + km} t^m.$$

Proof. We calculate using (2.1)

$$\begin{aligned}
(tq)_\infty \sum_{l=0}^{\infty} \frac{q^l}{(q)_l (tq)_{l+k}} &= \sum_{l=0}^{\infty} \frac{q^l (tq^{l+1+k})_\infty}{(q)_l} \\
&= \sum_{l=0}^{\infty} \frac{q^l}{(q)_l} \sum_{m=0}^{\infty} \frac{(-tq^{l+1+k})^m q^{m(m-1)/2}}{(q)_m} \\
&= \sum_{m=0}^{\infty} \frac{(-t)^m q^{m(m+1)/2+mk}}{(q)_m} \sum_{l=0}^{\infty} \frac{q^{(m+1)l}}{(q)_l} \\
&= \sum_{m=0}^{\infty} \frac{(-t)^m q^{m(m+1)/2+mk}}{(q)_m} \frac{1}{(q^{m+1})_\infty} \\
&= \frac{1}{(q)_\infty} \sum_{m \geq 0} (-1)^m q^{m(m+1)/2+mk} t^m.
\end{aligned}$$

This finishes the proof. \square

We give a short elementary proof of the following identity that appeared in the literature with different but more complicated proofs (cf. e.g. [BCMN, FeF, K1]).

Theorem 2.1. *We have*

$$\begin{aligned}
\frac{1}{(u)_\infty (u^{-1}q)_\infty} &= \frac{1}{(q)_\infty^2} \sum_{m=0}^{\infty} (-1)^m q^{\frac{1}{2}m(m+1)} \left(\sum_{k \geq 0} q^{km} u^k + \sum_{k > 0} q^{k(m+1)} u^{-k} \right) \\
&= \frac{1}{(q)_\infty^2} \sum_{m \in \mathbb{Z}} (-1)^m q^{\frac{1}{2}m(m+1)} \frac{1}{1 - uq^m}.
\end{aligned}$$

Here it is understood that

$$(2.3) \quad \frac{1}{1 - uq^m} = \begin{cases} \sum_{k=0}^{\infty} (uq^m)^k, & \text{if } m \geq 0, \\ -\sum_{k=1}^{\infty} (u^{-1}q^{-m})^k, & \text{if } m < 0. \end{cases}$$

Proof. Using Proposition 2.1 (in the fourth line below) and (2.1), we have

$$\begin{aligned}
&\frac{1}{(u)_\infty (u^{-1}q)_\infty} \\
&= \sum_{l=0}^{\infty} \frac{u^l}{(q)_l} \sum_{n=0}^{\infty} \frac{(u^{-1}q)^n}{(q)_n} \\
&= \sum_{k \geq 0} u^k \sum_{n=0}^{\infty} \frac{q^n}{(q)_n (q)_{n+k}} + \sum_{k > 0} u^{-k} \sum_{n=0}^{\infty} \frac{q^{n+k}}{(q)_n (q)_{n+k}} \\
&= \frac{1}{(q)_\infty^2} \left(\sum_{k \geq 0} \sum_{m=0}^{\infty} (-1)^m q^{m(m+1)/2} q^{km} u^k + \sum_{k > 0} \sum_{m=0}^{\infty} (-1)^m q^{m(m+1)/2} q^{km} q^k u^{-k} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(q)_\infty^2} \sum_{m=0}^{\infty} (-1)^m q^{\frac{1}{2}m(m+1)} \left(\sum_{k \geq 0} q^{km} u^k + \sum_{k > 0} q^{k(m+1)} u^{-k} \right) \\
&= \frac{1}{(q)_\infty^2} \sum_{m \in \mathbb{Z}} (-1)^m q^{\frac{1}{2}m(m+1)} \frac{1}{1 - uq^m}.
\end{aligned}$$

The last equation follows by the interpretation (2.3). \square

2.2. Lie algebra a_∞ . Denote by \mathfrak{gl} the Lie algebra of all matrices $(a_{ij})_{i,j \in \mathbb{Z}}$ satisfying $a_{ij} = 0$ for $|i - j| \gg 0$. Denote by E_{ij} the infinite matrix with 1 at $(i, j)^{th}$ place and 0 elsewhere and let the weight of E_{ij} be $j - i$. This defines a \mathbb{Z} -principal gradation $\mathfrak{gl} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{gl}_j$. Denote by $a_\infty \equiv \widehat{\mathfrak{gl}}_\infty = \mathfrak{gl} \oplus \mathbb{C}C$ the central extension given by the following 2-cocycle with values in \mathbb{C} (cf. [DJKM1]):

$$(2.4) \quad \alpha(A, B) = \text{tr}([J, A]B),$$

where $J = \sum_{j \leq 0} E_{jj}$. The \mathbb{Z} -gradation of the Lie algebra \mathfrak{gl} extends to a_∞ by letting the weight of C be 0. This leads to a triangular decomposition (i.e. a direct sum of subspaces of positive, zero, and negative weights):

$$a_\infty = (a_\infty)_+ \oplus (a_\infty)_0 \oplus (a_\infty)_-,$$

where $(a_\infty)_0 = \mathfrak{gl}_0 \oplus \mathbb{C}C$. Let

$$H_i^a = E_{ii} - E_{i+1, i+1} + \delta_{i,0}C, \quad i \in \mathbb{Z}.$$

Denote by $L(a_\infty; \Lambda, k)$ the highest weight a_∞ -module with highest weight $\Lambda \in (a_\infty)_0^*$ and level k , where C acts as a scalar $k \cdot I$. Let $\Lambda_j^a \in (a_\infty)_0^*$ be the fundamental weights, i.e. $\Lambda_j^a(H_i^a) = \delta_{ij}$. The Dynkin diagram for a_∞ , with fundamental weights labeled, is the following:

$$\cdots \circ \text{---} \underset{-2}{\circ} \text{---} \underset{-1}{\circ} \text{---} \underset{0}{\circ} \text{---} \underset{1}{\circ} \text{---} \underset{2}{\circ} \cdots$$

2.3. The 1-point a_∞ -functions of level -1 . Consider a pair of free bosonic ghosts

$$\gamma^\pm(z) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} \gamma_r^\pm z^{-r - \frac{1}{2}},$$

with nontrivial commutation relations

$$[\gamma_r^+, \gamma_s^-] = \delta_{r+s, 0}, \quad r, s \in \frac{1}{2} + \mathbb{Z}.$$

Let \mathfrak{F}^{-1} denote the Fock space generated by $\gamma^\pm(z)$ with vacuum vector $|0\rangle$ (cf. [K1] for more on Fock spaces and normal ordering $: \cdot :$ in vertex algebras). An

action of a_∞ of level -1 on \mathfrak{F}^{-1} is given by (cf. e.g. [W1]):

$$E(z, w) \equiv \sum_{i, j \in \mathbb{Z}} E_{ij} z^{i-1} w^{-j} = -:\gamma^+(z)\gamma^-(w):,$$

so that $E_{ij} = -:\gamma_{-i+\frac{1}{2}}^+ \gamma_{j-\frac{1}{2}}^-:$. The Virasoro field is given by

$$L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = \frac{1}{2} (: \gamma^+(z) \partial \gamma^-(z) : - : \partial \gamma^+(z) \gamma^-(z) :),$$

and we have $[L_0, \gamma_r^\pm] = -r \gamma_r^\pm$. According to the eigenvalues of the charge operator $e_{11} = \sum_{r \in \frac{1}{2} + \mathbb{Z}} :\gamma_r^+ \gamma_{-r}^-:$, the a_∞ -module \mathfrak{F}^{-1} has the following decomposition:

$$\mathfrak{F}^{-1} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{F}_{(n)}^{-1}.$$

Following Bloch-Okounkov [BO], we introduce the following operators in a_∞ :

$$\begin{aligned} :A(t): &= \sum_{k \in \mathbb{Z}} E_{kk} t^{k-\frac{1}{2}}, \\ A(t) &= :A(t): + \frac{C}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}. \end{aligned}$$

When acting on \mathfrak{F}^{-1} , $A(t) = -\sum_{r \in \frac{1}{2} + \mathbb{Z}} t^r \gamma_{-r}^+ \gamma_r^-$.

Define the Bloch-Okounkov n -point a_∞ -correlation function (or n -point a_∞ -function for short) of level -1 (associated to $m \in \mathbb{Z}$) to be

$$\mathfrak{A}_{-1}^{(m)}(q; t_1, \dots, t_n) := \text{tr}_{\mathfrak{F}_{(m)}^{-1}}(q^{L_0} A(t_1) \cdots A(t_n)).$$

For a partition λ we use $\ell(\lambda)$ and $|\lambda|$ to denote the length and the size of λ respectively.

Lemma 2.1. *We have*

$$\begin{aligned} \text{(i)} \quad \sum_{\ell(\lambda)=l} q^{|\lambda|} &= \frac{q^l}{(q)_l}, \\ \text{(ii)} \quad \sum_{\ell(\lambda)=l} q^{|\lambda|} t^{\lambda_i} &= \frac{t q^l}{(1-q) \cdots (1-q^{i-1})(1-q^i t) \cdots (1-q^l t)}, \quad i \leq l. \end{aligned}$$

Proof. Part (i) follows from the identity $\sum_{\ell(\lambda) \leq l} q^{|\lambda|} = (q)_l^{-1}$, while (ii) follows from another well-known identity

$$\sum_{\ell(\lambda) \leq l} q^{|\lambda|} t^{\lambda_i} = \frac{1}{(1-q) \cdots (1-q^{i-1})(1-q^i t) \cdots (1-q^l t)}.$$

□

Theorem 2.2. *The one-point function $\mathfrak{A}_{-1}^{(0)}(q; t)$ is given by*

$$\begin{aligned} & \frac{{}_2\Phi_1(0, 0; q; q)}{t^{-\frac{1}{2}} - t^{\frac{1}{2}}} + t^{\frac{1}{2}} \sum_{i=1}^{\infty} \frac{q^{i-1}}{(q)_{i-1}(q)_{i-1}} ({}_3\Phi_2(0, 0, q; tq^i, q^i; q) - 1) \\ & - t^{-\frac{1}{2}} \sum_{i=1}^{\infty} \frac{q^{i-1}}{(q)_{i-1}(q)_{i-1}} ({}_3\Phi_2(0, 0, q; t^{-1}q^i, q^i; q) - 1). \end{aligned}$$

Proof. Note that $\gamma_{-r_1}^+ \gamma_{-r_2}^+ \cdots \gamma_{-r_l}^+ \gamma_{-s_1}^- \cdots \gamma_{-s_l}^- |0\rangle$ is an eigenvector of $q^{L_0} \mathbf{A}(t)$ of eigenvalue $q^{\sum_{i=1}^l (r_i + s_i)} \left(\sum_{i=1}^l (t^{r_i} - t^{-s_i}) - \sum_{r \in \frac{1}{2} + \mathbb{Z}_+} t^{-r} \right)$. Since $\mathfrak{F}_{(0)}^{-1}$ has a basis given by $\gamma_{-r_1}^+ \cdots \gamma_{-r_l}^+ \gamma_{-s_1}^- \cdots \gamma_{-s_l}^- |0\rangle$ for $r_1 \geq \cdots \geq r_l > 0; s_1 \geq \cdots \geq s_l > 0, l \geq 0$, we have by Lemma 2.1 that

$$\begin{aligned} (2.5) \quad \mathfrak{A}_{-1}^{(0)}(q; t) &= \sum_{l=0}^{\infty} q^{-l} \sum_{\ell(\lambda)=\ell(\mu)=l} q^{|\lambda|+|\mu|} \left(\sum_{i=1}^l (t^{\lambda_i - \frac{1}{2}} - t^{-\mu_i + \frac{1}{2}}) + \frac{1}{t^{-\frac{1}{2}} - t^{\frac{1}{2}}} \right) \\ &= \frac{{}_2\Phi_1(0, 0; q; q)}{t^{-\frac{1}{2}} - t^{\frac{1}{2}}} + \sum_{l=1}^{\infty} q^{-l} \sum_{\ell(\lambda)=\ell(\mu)=l} q^{|\lambda|+|\mu|} \left(\sum_{i=1}^l (t^{\lambda_i - \frac{1}{2}} - t^{-\mu_i + \frac{1}{2}}) \right). \end{aligned}$$

We compute that

$$\begin{aligned} & \sum_{l=1}^{\infty} q^{-l} \sum_{\ell(\lambda)=\ell(\mu)=l} q^{|\lambda|+|\mu|} \sum_{i=1}^l t^{\lambda_i - \frac{1}{2}} \\ &= t^{-\frac{1}{2}} \sum_{l=1}^{\infty} q^{-l} \sum_{i=1}^l \frac{q^{2l} t}{(1-q) \cdots (1-q^{i-1})(1-q^i t) \cdots (1-q^l t)(q)_l} \\ &= t^{\frac{1}{2}} \sum_{i=1}^{\infty} \sum_{l=i}^{\infty} \frac{q^l}{(q)_{i-1}(q)_l (q^i t)_{l-i+1}} \\ &= t^{\frac{1}{2}} \sum_{i=1}^{\infty} \frac{q^{i-1}}{(q)_{i-1}(q)_{i-1}} \sum_{s=1}^{\infty} \frac{q^s}{(tq^i)_s (q^i)_s} \\ &= t^{\frac{1}{2}} \sum_{i=1}^{\infty} \frac{q^{i-1}}{(q)_{i-1}(q)_{i-1}} ({}_3\Phi_2(0, 0, q; tq^i, q^i; q) - 1). \end{aligned}$$

Now the theorem follows from this computation and (2.5). \square

2.4. The generalized 1-point a_{∞} -function. Let A be the operator on \mathfrak{F}^{-1} acting trivially on $|0\rangle$ such that $[A, \gamma_r^+] = \gamma_r^+$, and $[A, \gamma_r^-] = 0$, for all $r \in \frac{1}{2} + \mathbb{Z}$. Let B be the operator acting trivially on $|0\rangle$ with $[B, \gamma_r^-] = \gamma_r^-$, and $[B, \gamma_r^+] = 0$, for all $r \in \frac{1}{2} + \mathbb{Z}$.

Theorem 2.3. *We have*

$$\begin{aligned} \text{tr}_{\mathfrak{F}^{-1}}(q^{L_0} x^A y^B \mathbf{A}(t)) &= \frac{1}{(t^{-\frac{1}{2}} - t^{\frac{1}{2}})(xq^{\frac{1}{2}})_\infty (yq^{\frac{1}{2}})_\infty} \\ &+ \frac{x(tq)^{\frac{1}{2}}(xtq^{\frac{3}{2}})_\infty}{(1 - xq^{\frac{1}{2}})(tq)_\infty (xq^{\frac{1}{2}})_\infty (yq^{\frac{1}{2}})_\infty} {}_2\Phi_2(xq^{\frac{1}{2}}, xq^{\frac{1}{2}}; xq^{\frac{3}{2}}, xtq^{\frac{3}{2}}; tq^2) \\ &- \frac{y(t^{-1}q)^{\frac{1}{2}}(yt^{-1}q^{\frac{3}{2}})_\infty}{(1 - yq^{\frac{1}{2}})(t^{-1}q)_\infty (xq^{\frac{1}{2}})_\infty (yq^{\frac{1}{2}})_\infty} {}_2\Phi_2(yq^{\frac{1}{2}}, yq^{\frac{1}{2}}; yq^{\frac{3}{2}}, yt^{-1}q^{\frac{3}{2}}; t^{-1}q^2). \end{aligned}$$

Proof. By Lemma 2.1 and (2.1), we have

$$\sum_{k=0}^{\infty} q^{-\frac{k}{2}} z^k \sum_{\ell(\mu)=k} q^{|\mu|} = \frac{1}{(zq^{\frac{1}{2}})_\infty}.$$

Since the vector $\gamma_{-r_1}^+ \gamma_{-r_2}^+ \cdots \gamma_{-r_l}^+ \gamma_{-s_1}^- \cdots \gamma_{-s_k}^- |0\rangle$ is an eigenvector of $x^A y^B q^{L_0} \mathbf{A}(t)$ of eigenvalue $x^l y^k q^{\sum_{i=1}^l r_i + \sum_{j=1}^k s_j} \left(\sum_{i=1}^l t^{r_i} - \sum_{j=1}^k t^{-s_j} - \sum_{r \in \frac{1}{2} + \mathbb{Z}_+} t^{-r} \right)$, we have

$$\begin{aligned} \text{tr}_{\mathfrak{F}^{-1}}(x^A y^B q^{L_0} \mathbf{A}(t)) &= \sum_{k=0}^{\infty} y^k q^{-\frac{k}{2}} \sum_{\ell(\mu)=k} q^{|\mu|} \sum_{l=0}^{\infty} x^l q^{-\frac{l}{2}} \sum_{\ell(\lambda)=l} q^{|\lambda|} \left(\sum_{i=1}^l t^{\lambda_i - \frac{1}{2}} - \sum_{j=1}^k t^{-\mu_j + \frac{1}{2}} \right) \\ &+ \frac{1}{(xq^{\frac{1}{2}})_\infty (yq^{\frac{1}{2}})_\infty (t^{-\frac{1}{2}} - t^{\frac{1}{2}})}. \end{aligned}$$

Next we claim that

$$\begin{aligned} (2.6) \quad &\sum_{l=1}^{\infty} x^l q^{-\frac{l}{2}} \sum_{\ell(\lambda)=l} q^{|\lambda|} \sum_{i=1}^l t^{\lambda_i - \frac{1}{2}} \\ &= \frac{x(tq)^{\frac{1}{2}}(xtq^{\frac{3}{2}})_\infty}{(1 - xq^{\frac{1}{2}})(tq)_\infty (xq^{\frac{1}{2}})_\infty} {}_2\Phi_2(xq^{\frac{1}{2}}, xq^{\frac{1}{2}}; xq^{\frac{3}{2}}, xtq^{\frac{3}{2}}; tq^2). \end{aligned}$$

To see this we compute directly

$$\begin{aligned} &\sum_{l=1}^{\infty} x^l q^{-\frac{l}{2}} \sum_{\ell(\lambda)=l} q^{|\lambda|} \sum_{i=1}^l t^{\lambda_i - \frac{1}{2}} \\ &= t^{\frac{1}{2}} \sum_{l=1}^{\infty} \sum_{i=1}^{\infty} \frac{x^l q^{\frac{l}{2}}}{(q)_{i-1} (1 - tq^i) \cdots (1 - tq^l)} \\ &= t^{\frac{1}{2}} \sum_{i=1}^{\infty} \frac{1}{(q)_{i-1} (tq^i)_\infty} \sum_{l=i}^{\infty} x^l q^{\frac{l}{2}} (tq^{l+1})_\infty \end{aligned}$$

$$\begin{aligned}
&= t^{\frac{1}{2}} \sum_{i=1}^{\infty} \frac{1}{(q)_{i-1}(tq^i)_{\infty}} \sum_{l=i}^{\infty} x^l q^{\frac{l}{2}} \sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}m(m-1)} q^{m(l+1)} (-t)^m}{(q)_m} \\
&= t^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}m(m-1)} (-tq)^m}{(q)_m} \sum_{i=1}^{\infty} \frac{1}{(q)_{i-1}(tq^i)_{\infty}} \sum_{l=i}^{\infty} x^l q^{l(m+\frac{1}{2})} \\
&= \frac{t^{\frac{1}{2}}}{(tq)_{\infty}} \sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}m(m-1)} (-tq)^m}{(q)_m} \sum_{i=1}^{\infty} \frac{(tq)_{i-1}}{(q)_{i-1}} \frac{x^i q^{i(m+\frac{1}{2})}}{1 - xq^{m+\frac{1}{2}}} \\
&= \frac{xt^{\frac{1}{2}}}{(tq)_{\infty}} \sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}m(m-1)} (-tq)^m q^{m+\frac{1}{2}}}{(q)_m (1 - xq^{m+\frac{1}{2}})} \sum_{i=1}^{\infty} \frac{(tq)_{i-1}}{(q)_{i-1}} (xq^{m+\frac{1}{2}})^{i-1} \\
&= \frac{xt^{\frac{1}{2}}}{(1 - xq^{\frac{1}{2}})(tq)_{\infty}} \sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}m(m-1)} (-tq)^m q^{m+\frac{1}{2}} (xq^{\frac{1}{2}})_m (xtq^{m+\frac{3}{2}})_{\infty}}{(q)_m (xq^{\frac{3}{2}})_m (xq^{m+\frac{1}{2}})_{\infty}} \\
&= \frac{x(tq)^{\frac{1}{2}} (xtq^{\frac{3}{2}})_{\infty}}{(1 - xq^{\frac{1}{2}})(tq)_{\infty} (xq^{\frac{1}{2}})_{\infty}} \sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}m(m-1)} (-tq^2)^m (xq^{\frac{1}{2}})_m^2}{(q)_m (xq^{\frac{3}{2}})_m (xtq^{\frac{3}{2}})_m} \\
&= \frac{x(tq)^{\frac{1}{2}} (xtq^{\frac{3}{2}})_{\infty}}{(1 - xq^{\frac{1}{2}})(tq)_{\infty} (xq^{\frac{1}{2}})_{\infty}} {}_2\Phi_2(xq^{\frac{1}{2}}, xq^{\frac{1}{2}}; xq^{\frac{3}{2}}, xtq^{\frac{3}{2}}; tq^2).
\end{aligned}$$

The theorem now follows from this and a similar expression for the other summation (involving y^B). \square

Remark 2.1. Noting that the operator $B - A$ is the same as the operator e_{11} defined earlier, we have

$$\mathfrak{A}_{-1}^{(m)}(q; t) = [z^m] \operatorname{tr}_{\mathfrak{F}^{-1}}(q^{L_0} x^A y^B \mathbf{A}(t))|_{x=z^{-1}, y=z}.$$

Here and below $[z^m]X$ denotes the coefficient of z^m in the expansion of X in z . For this reason, we refer to $\operatorname{tr}_{\mathfrak{F}^{-1}}(q^{L_0} x^A y^B \mathbf{A}(t))$ as the generalized 1-point a_{∞} -function.

2.5. The generalized 2-point a_{∞} -function. We will compute the generalized 2-point a_{∞} -function $\operatorname{tr}_{\mathfrak{F}^{-1}} q^{L_0} x^A y^B \mathbf{A}(t_1) \mathbf{A}(t_2)$. Similar to the computation of the generalized 1-point function in the previous subsection, the calculation of the generalized 2-point function essentially boils down to the following calculation:

$$\begin{aligned}
&\sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \sum_{l=j}^{\infty} \frac{x^l q^{\frac{l}{2}} t_1 t_2}{(q)_{i-1} (1 - q^i t_1) \cdots (1 - q^{j-1} t_1) (1 - q^j t_1 t_2) \cdots (1 - q^l t_1 t_2)} \\
&= \frac{t_1 t_2}{(qt_1 t_2)_{\infty}} \sum_{i=1}^{\infty} \frac{1}{(q)_{i-1}} \sum_{j=i+1}^{\infty} \frac{(qt_1 t_2)_{j-1}}{(1 - q^i t_1) \cdots (1 - q^{j-1} t_1)} \sum_{l=j}^{\infty} x^l q^{\frac{l}{2}} (q^{l+1} t_1 t_2)_{\infty}
\end{aligned}$$

$$\begin{aligned}
&= \frac{t_1 t_2}{(qt_1)_\infty (qt_1 t_2)_\infty} \sum_{i=1}^{\infty} \frac{(qt_1)_{i-1}}{(q)_{i-1}} \sum_{j=i+1}^{\infty} (q^j t_1)_\infty (qt_1 t_2)_{j-1} \sum_{l=j}^{\infty} x^l q^{\frac{l}{2}} (q^{l+1} t_1 t_2)_\infty \\
&= \frac{t_1 t_2}{(qt_1)_\infty (qt_1 t_2)_\infty} \sum_{i=1}^{\infty} \frac{(qt_1)_{i-1}}{(q)_{i-1}} \sum_{j=i+1}^{\infty} (q^j t_1)_\infty (qt_1 t_2)_{j-1} \\
&\quad \times \sum_{s=0}^{\infty} \frac{(-qt_1 t_2)^s q^{\frac{1}{2}s(s-1)}}{(q)_s} \sum_{l=j}^{\infty} (xq^{s+\frac{1}{2}})^l \\
&= \frac{t_1 t_2}{(qt_1)_\infty (qt_1 t_2)_\infty} \\
&\quad \times \sum_{i=1}^{\infty} \frac{(qt_1)_{i-1}}{(q)_{i-1}} \sum_{j=i+1}^{\infty} (q^j t_1)_\infty (qt_1 t_2)_{j-1} \sum_{s=0}^{\infty} \frac{(-qt_1 t_2)^s q^{\frac{1}{2}s(s-1)} (xq^{s+\frac{1}{2}})^j}{(q)_s (1 - xq^{s+\frac{1}{2}})} \\
&= \frac{t_1 t_2}{(qt_1)_\infty (qt_1 t_2)_\infty} \\
&\quad \times \sum_{i=1}^{\infty} \frac{(qt_1)_{i-1}}{(q)_{i-1}} \sum_{s=0}^{\infty} \frac{(-qt_1 t_2)^s q^{\frac{1}{2}s(s-1)}}{(q)_s (1 - xq^{s+\frac{1}{2}})} \sum_{j=i+1}^{\infty} (xq^{s+\frac{1}{2}})^j (q^j t_1)_\infty (qt_1 t_2)_{j-1},
\end{aligned}$$

which is equal to

$$\begin{aligned}
&\frac{t_1 t_2}{(qt_1)_\infty} \sum_{i=1}^{\infty} \frac{(qt_1)_{i-1}}{(q)_{i-1}} \sum_{s=0}^{\infty} \frac{(-qt_1 t_2)^s q^{\frac{1}{2}s(s-1)}}{(q)_s (1 - xq^{s+\frac{1}{2}})} \sum_{j=i+1}^{\infty} \frac{(q^j t_1)_\infty}{(q^j t_1 t_2)_\infty} (xq^{s+\frac{1}{2}})^j \\
&= \frac{t_1 t_2}{(qt_1)_\infty} \sum_{i=1}^{\infty} \frac{(qt_1)_{i-1}}{(q)_{i-1}} \sum_{s=0}^{\infty} \frac{(-qt_1 t_2)^s q^{\frac{1}{2}s(s-1)}}{(q)_s (1 - xq^{s+\frac{1}{2}})} \sum_{j=i+1}^{\infty} \sum_{m=0}^{\infty} \frac{(t_2^{-1})_m}{(q)_m} (q^j t_1 t_2)^m (xq^{s+\frac{1}{2}})^j \\
&= \frac{t_1 t_2}{(qt_1)_\infty} \sum_{i=1}^{\infty} \frac{(qt_1)_{i-1}}{(q)_{i-1}} \sum_{s=0}^{\infty} \frac{(-qt_1 t_2)^s q^{\frac{1}{2}s(s-1)}}{(q)_s (1 - xq^{s+\frac{1}{2}})} \sum_{m=0}^{\infty} \frac{(t_2^{-1})_m}{(q)_m} (t_1 t_2)^m \sum_{j=i+1}^{\infty} (xq^{m+s+\frac{1}{2}})^j \\
&= \frac{t_1 t_2}{(qt_1)_\infty} \sum_{i=1}^{\infty} \frac{(qt_1)_{i-1}}{(q)_{i-1}} \sum_{s=0}^{\infty} \frac{(-qt_1 t_2)^s q^{\frac{1}{2}s(s-1)}}{(q)_s (1 - xq^{s+\frac{1}{2}})} \sum_{m=0}^{\infty} \frac{(t_2^{-1})_m}{(q)_m} (t_1 t_2)^m \frac{(xq^{m+s+\frac{1}{2}})^{i+1}}{1 - xq^{m+s+\frac{1}{2}}} \\
&= \frac{t_1 t_2}{(qt_1)_\infty} \sum_{s=0}^{\infty} \frac{(-qt_1 t_2)^s q^{\frac{1}{2}s(s-1)}}{(q)_s (1 - xq^{s+\frac{1}{2}})} \\
&\quad \times \sum_{m=0}^{\infty} \frac{(t_2^{-1})_m}{(q)_m} (t_1 t_2)^m \frac{(xq^{m+s+\frac{1}{2}})^2}{1 - xq^{m+s+\frac{1}{2}}} \sum_{i=1}^{\infty} \frac{(qt_1)_{i-1}}{(q)_{i-1}} (xq^{m+s+\frac{1}{2}})^{i-1} \\
&= \frac{t_1 t_2}{(qt_1)_\infty} \sum_{s=0}^{\infty} \frac{(-qt_1 t_2)^s q^{\frac{1}{2}s(s-1)}}{(q)_s (1 - xq^{s+\frac{1}{2}})} \sum_{m=0}^{\infty} \frac{(t_2^{-1})_m}{(q)_m} (t_1 t_2)^m \frac{(xq^{m+s+\frac{1}{2}})^2}{1 - xq^{m+s+\frac{1}{2}}} \frac{(xq^{m+s+\frac{3}{2}} t_1)_\infty}{(xq^{m+s+\frac{1}{2}})_\infty}
\end{aligned}$$

$$\begin{aligned}
&= \frac{x^2 q t_1 t_2 (x q^{\frac{3}{2}} t_1)_\infty}{(1 - x q^{\frac{1}{2}})^2 (x q^{\frac{1}{2}})_\infty (q t_1)_\infty} \\
&\quad \times \sum_{s=0}^{\infty} \frac{(x q^{\frac{1}{2}})_s^3 (-q^3 t_1 t_2)^s q^{\frac{1}{2}s(s-1)}}{(x q^{\frac{3}{2}} t_1)_s (q)_s (x q^{\frac{3}{2}})_s^2} \sum_{m=0}^{\infty} \frac{(t_2^{-1})_m (q^2 t_1 t_2)^m (x q^{s+\frac{1}{2}})_m^2}{(q)_m (x q^{s+\frac{3}{2}})_m (x q^{s+\frac{3}{2}} t_1)_m} \\
&= \frac{x^2 q t_1 t_2 (x q^{\frac{3}{2}} t_1)_\infty}{(1 - x q^{\frac{1}{2}})^2 (x q^{\frac{1}{2}})_\infty (q t_1)_\infty} \\
&\quad \times \sum_{s=0}^{\infty} \frac{(x q^{\frac{1}{2}})_s^3 (-q^3 t_1 t_2)^s q^{\frac{1}{2}s(s-1)}}{(x q^{\frac{3}{2}} t_1)_s (q)_s (x q^{\frac{3}{2}})_s^2} {}_3\Phi_2(t_2^{-1}, x q^{s+\frac{1}{2}}, x q^{s+\frac{1}{2}}; x q^{s+\frac{3}{2}}, x q^{s+\frac{3}{2}} t_1; q^2 t_1 t_2).
\end{aligned}$$

Denote the above expression by $\bar{\Gamma}(x, t_1, t_2)$. We point out that all the 3-2 q -hypergeometric series in $\bar{\Gamma}(x, t_1, t_2)$ are of type II, meaning that ${}_3\Phi_2(a, b, c; d, e; z)$ satisfies $\frac{de}{abc} = z$. Define

$$\Gamma(x, y, t_1, t_2) := \frac{(t_1 t_2)^{-\frac{1}{2}}}{(q^{\frac{1}{2}} y)_\infty} (\bar{\Gamma}(x, t_1, t_2) + \bar{\Gamma}(x, t_2, t_1)).$$

To simplify notation we set

$$\Omega(x, y, t) := \frac{x(tq)^{\frac{1}{2}}(xtq^{\frac{3}{2}})_\infty}{(1 - xq^{\frac{1}{2}})(tq)_\infty(xq^{\frac{1}{2}})_\infty(yq^{\frac{1}{2}})_\infty} {}_2\Phi_2(xq^{\frac{1}{2}}, xq^{\frac{1}{2}}; xq^{\frac{3}{2}}, txq^{\frac{3}{2}}; tq^2).$$

The following is now a straightforward calculation, based on the above calculations.

Theorem 2.4. *The generalized 2-point a_∞ -function $\text{tr}_{\mathfrak{F}^{-1}} q^{L_0} x^A y^B \mathbf{A}(t_1) \mathbf{A}(t_2)$ is equal to*

$$\begin{aligned}
&\Gamma(x, y, t_1, t_2) + \Gamma(y, x, t_1^{-1}, t_2^{-1}) + \Omega(x, y, t_1 t_2) + \Omega(y, x, t_1^{-1} t_2^{-1}) \\
&+ \frac{1}{t_1^{-\frac{1}{2}} - t_1^{\frac{1}{2}}} (\Omega(x, y, t_2) - \Omega(y, x, t_2^{-1})) + \frac{1}{t_2^{-\frac{1}{2}} - t_2^{\frac{1}{2}}} (\Omega(x, y, t_1) - \Omega(y, x, t_1^{-1})) \\
&- (x q^{\frac{1}{2}})_\infty (y q^{\frac{1}{2}})_\infty (\Omega(x, y, t_1) \Omega(y, x, t_2^{-1}) + \Omega(x, y, t_1^{-1}) \Omega(y, x, t_2)) \\
&+ \frac{1}{(t_1^{-\frac{1}{2}} - t_1^{\frac{1}{2}})(t_2^{-\frac{1}{2}} - t_2^{\frac{1}{2}})(x q^{\frac{1}{2}})_\infty (y q^{\frac{1}{2}})_\infty}.
\end{aligned}$$

Remark 2.2. The 2-point a_∞ -function can be recovered from the generalized 2-point a_∞ -function:

$$\mathfrak{A}_{-1}^{(m)}(q; t_1, t_2) = [z^m] \text{tr}_{\mathfrak{F}^{-1}} (q^{L_0} x^A y^B \mathbf{A}(t_1) \mathbf{A}(t_2))|_{x=z^{-1}, y=z}.$$

2.6. The n -point a_∞ -functions of level $-l$. Let $l \in \mathbb{N}$. Recall a generalized partition λ of depth l is an ordered l -tuple of non-increasing integers; that is, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ where $\lambda_1 \geq \dots \geq \lambda_l$. Associated to a generalized partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ we define a highest weight $\Lambda(\lambda)$ of a_∞ via

$$\Lambda(\lambda) = (\lambda_l - \lambda_1 - l)\Lambda_0^a + \sum_{k=1}^{i-1} (\lambda_k - \lambda_{k+1})\Lambda_l^a + \lambda_i\Lambda_i^a,$$

where i is the index of the last positive entry of λ .

We let $\mathfrak{F}^{-l} := (\mathfrak{F}^{-1})^{\otimes l}$ to denote the Fock space generated by l pairs of free bosonic fields $\gamma^{\pm, i}(z) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} \gamma_r^{\pm, i} z^{-r-1/2}$, $i = 1, \dots, l$, with non-trivial commutation relations $[\gamma_r^{+, i}, \gamma_s^{-, j}] = \delta_{ij} \delta_{r+s, 0}$. On \mathfrak{F}^{-l} there is an action of a_∞ of level $-l$ given by

$$E(z, w) = - \sum_{i=1}^l : \gamma^{i, +}(z) \gamma^{i, -}(w) :.$$

Furthermore there exists a commuting action of the general linear algebra $\mathfrak{gl}(l)$ whose elementary matrices e_{ij} acts by the formula

$$e_{ij} = - \sum_{r \in 1/2 + \mathbb{Z}} : \gamma_{-r}^{+, i} \gamma_r^{-, j} :.$$

This action lifts to that of $GL(l)$.

Proposition 2.2. [KR] (also see [W1, Theorem 5.1]) *We have the following $(GL(l), a_\infty)$ -module decomposition:*

$$\mathfrak{F}^{-l} \cong \bigoplus_{\lambda} V_{\lambda}(GL(l)) \otimes L(a_\infty; \Lambda(\lambda), -l),$$

where λ is a generalized partition of depth l and $V_{\lambda}(GL(l))$ is the irreducible $GL(l)$ -module of highest weight λ .

The Bloch-Okounkov n -point a_∞ -function of level $-l$ (associated to a generalized partition λ of depth l) is defined as

$$\mathfrak{A}_{-l}^{\lambda}(q; t_1, \dots, t_n) = \text{tr}_{L(a_\infty; \Lambda(\lambda), -l)} q^{L_0} \mathbf{A}(t_1) \mathbf{A}(t_2) \cdots \mathbf{A}(t_n).$$

Theorem 2.5. *The n -point a_∞ -function of level $-l$, $\mathfrak{A}_{-l}^{\lambda}(q; t_1, \dots, t_n)$, is equal to*

$$\sum_{\sigma \in S_l} (-1)^{\ell(\sigma)} \mathfrak{A}_{-1}^{(k_1)}(q; t_1, \dots, t_n) \cdots \mathfrak{A}_{-1}^{(k_l)}(q; t_1, \dots, t_n),$$

where $k_i = (\lambda + \rho - \sigma(\rho), \varepsilon_i)$.

Proof. We shall denote by e_{ij} , $1 \leq i, j \leq l$, the standard matrix elements in the Lie algebra $\mathfrak{gl}(l)$. Applying $\text{tr}_{\mathfrak{F}^{-l}} z_1^{e_{11}} \cdots z_l^{e_{ll}} q^{L_0} \mathbf{A}(t_1) \cdots \mathbf{A}(t_n)$ to both sides of the identity in Proposition 2.2, we obtain that

$$\prod_{i=1}^l \text{tr}_{\mathfrak{F}^{-1}} z_i^{e_{ii}} q^{L_0} \mathbf{A}(t_1) \cdots \mathbf{A}(t_n) = \sum_{\lambda} \text{ch}_{\lambda}^{gl}(z_1, \dots, z_l) \mathfrak{A}_{-l}^{\lambda}(q; t_1, \dots, t_n),$$

since $z_i^{e_{ii}}$ acts on the left-hand side only on the i^{th} tensor factor. On the other hand, we have

$$\begin{aligned} \prod_{i=1}^l \text{tr}_{\mathfrak{F}^{-1}} z_i^{e_{ii}} q^{L_0} \mathbf{A}(t_1) \cdots \mathbf{A}(t_n) &= \prod_{i=1}^l \left(\sum_{m_i \in \mathbb{Z}} z_i^{m_i} \mathfrak{A}_{-1}^{(m_i)}(q; t_1, \dots, t_n) \right) \\ &= \sum_{\mathbf{m} \in \mathbb{Z}^l} z_1^{m_1} \cdots z_l^{m_l} \mathfrak{A}_{-1}^{(m_1)}(q; t_1, \dots, t_n) \cdots \mathfrak{A}_{-1}^{(m_l)}(q; t_1, \dots, t_n). \end{aligned}$$

Recall (cf. [FH, pp. 399]) that

$$(2.7) \quad \text{ch}_{\lambda}^{gl}(z_1, \dots, z_l) = \frac{|z_j^{\lambda_i + l - i}|}{|z_j^{l - i}|},$$

where $|a_{ij}|$ denotes the determinant of the matrix (a_{ij}) .

Combining the above identities with the Weyl denominator formula, we obtain that

$$\begin{aligned} \sum_{\lambda} |z_j^{\lambda_i + l - i}| \mathfrak{A}_{-l}^{\lambda}(q; t_1, \dots, t_n) \\ &= \sum_{\sigma \in S_l} (-1)^{\ell(\sigma)} \mathbf{z}^{\sigma(\rho)} \prod_{i=1}^l \text{tr}_{\mathfrak{F}^{-1}} z_i^{e_{ii}} q^{L_0} \mathbf{A}(t_1) \cdots \mathbf{A}(t_n) \\ &= \sum_{\sigma \in S_l} (-1)^{\ell(\sigma)} \mathbf{z}^{\sigma(\rho)} \sum_{\mathbf{m} \in \mathbb{Z}^l} z_1^{m_1} \cdots z_l^{m_l} \mathfrak{A}_{-1}^{(m_1)}(q; t_1, \dots, t_n) \cdots \mathfrak{A}_{-1}^{(m_l)}(q; t_1, \dots, t_n), \end{aligned}$$

where we denote $\mathbf{z}^{\sigma(\rho)} = \prod_{i=1}^l z_i^{(\sigma(\rho), \varepsilon_i)}$.

Now the theorem follows from comparing the coefficients of $\prod_{i=1}^l z_i^{\lambda_i + l - i}$ on both sides of the above identity. \square

Remark 2.3. Combining Theorem 2.5 with the calculation of the 1-point and 2-point functions of level -1 in the previous subsections, we have computed the 1-point and 2-point a_{∞} -functions of level $-l$.

2.7. A q -difference equation for a_{∞} -functions. Even though it is difficult to compute the n -point function in general, we have the following q -difference equation of level -1 which could be helpful (compare [BO, Ok] for a difference equation of level 1).

Theorem 2.6. *The n -point a_∞ -function of level -1 satisfies the following q -difference equation:*

$$\begin{aligned} \mathfrak{A}_{-1}^{(0)}(q; qt_1, \dots, t_n) \\ = \sum_{s=0}^{n-1} (-1)^{s+1} \sum_{1 < i_1 < \dots < i_s \leq n} \mathfrak{A}_{-1}^{(0)}(q; t_1 t_{i_1} \cdots t_{i_s}, \dots, \hat{t}_{i_1}, \dots, \hat{t}_{i_s}, \dots), \end{aligned}$$

where $\hat{}$ denotes a deleted term.

Proof. Applying the commutation relation for $A(t)$ and γ_r^- repeatedly yields

$$A(t_2) \cdots A(t_n) \gamma_r^- = \sum_{P \subset \{2, \dots, n\}} (-1)^{|P|} \left(\prod_{i \in P} t_i^r \right) \gamma_r^- \prod_{i \notin P} A(t_i).$$

Below we write $\text{tr} = \text{tr}_{\mathfrak{F}_{(0)}^{-1}}$. Taking trace of the above gives

$$\text{tr } q^{L_0} \gamma_{-r}^+ A(t_2) \cdots A(t_n) \gamma_r^- = \sum_{P \subset \{2, \dots, n\}} (-1)^{|P|} \left(\prod_{i \in P} t_i^r \right) \text{tr } q^{L_0} \gamma_{-r}^+ \gamma_r^- \prod_{i \notin P} A(t_i).$$

It follows from the commutation relation for L_0 that

$$\begin{aligned} -\text{tr } q^{L_0} \gamma_{-r}^+ A(t_2) \cdots A(t_n) \gamma_r^- &= -\text{tr } \gamma_r^- q^{L_0} \gamma_{-r}^+ A(t_2) \cdots A(t_n) \\ &= -q^r \text{tr } q^{L_0} \gamma_r^- \gamma_{-r}^+ A(t_2) \cdots A(t_n). \end{aligned}$$

Multiplying both sides of the above by t_1^r and summing over $r \in \frac{1}{2} + \mathbb{Z}$, we obtain by using $A(t) = -\sum_{r \in \frac{1}{2} + \mathbb{Z}} t^r \gamma_r^- \gamma_{-r}^+$ (which follows from the definition of $A(t)$ and commutation relations among γ_r^\pm) that

$$\begin{aligned} \text{tr } q^{L_0} A(qt_1) A(t_2) \cdots A(t_n) \\ = \sum_{s=0}^{n-1} (-1)^{s+1} \sum_{1 < i_1 < \dots < i_s \leq n} \mathfrak{A}_{-1}^{(0)}(q; t_1 t_{i_1} \cdots t_{i_s}, \dots, \hat{t}_{i_1}, \dots, \hat{t}_{i_s}, \dots). \end{aligned}$$

This finishes the proof. \square

2.8. The q -dimension formula of an a_∞ -module of level $-l$. Let $l \in \mathbb{N}$. We shall denote the q -dimension of the module $L(a_\infty; \Lambda(\lambda), -l)$ by

$$Q_{-l}^\lambda(q) := \text{tr}_{L(a_\infty; \Lambda(\lambda), -l)} q^{L_0}.$$

Lemma 2.2. *We have the following identity:*

$$\prod_{i=1}^l \frac{1}{(z_i q^{\frac{1}{2}})_\infty (z_i^{-1} q^{\frac{1}{2}})_\infty} = \sum_{\lambda} \text{ch}_\lambda^{g^l}(z_1, \dots, z_l) \cdot \text{tr}_{L(a_\infty; \Lambda(\lambda), -l)} q^{L_0}.$$

Proof. Apply $\text{tr}_{\mathfrak{F}^{-l}} q^{L_0} z_1^{e_{11}} \cdots z_l^{e_{ll}}$ to both sides of Proposition 2.2. From the formulas for e_{ii} in [W1], on the left-hand side, $z_i^{e_{ii}}$ only acts on the i^{th} tensor factor and the resulting formula follows from the structure of the bosonic Fock space \mathfrak{F}^{-1} . For the right-hand side, only $z_1^{e_{11}} \cdots z_l^{e_{ll}}$ acts on $V_\lambda(GL(l))$. \square

Proposition 2.3. *Let $k \in \mathbb{Z}$. The q -dimension of the irreducible a_∞ -module of highest weight $\Lambda(k)$ and level -1 is*

$$Q_{-1}^{(k)}(q) = \frac{1}{(q)_\infty^2} \sum_{m \geq 0} (-1)^m q^{\frac{1}{2}m(m+1) + |k|(m+\frac{1}{2})}.$$

Proof. By Lemma 2.2 and the identity in Theorem 2.1 with $u = zq^{\frac{1}{2}}$, we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} z^k Q_{-1}^{(k)}(q) &= \frac{1}{(zq^{\frac{1}{2}})_\infty (z^{-1}q^{\frac{1}{2}})_\infty} \\ &= \frac{1}{(q)_\infty^2} \sum_{m=0}^{\infty} (-1)^m q^{\frac{1}{2}m(m+1)} \left(\sum_{k \geq 0} q^{k(m+\frac{1}{2})} z^k + \sum_{k > 0} q^{k(m+\frac{1}{2})} z^{-k} \right) \\ &= \frac{1}{(q)_\infty^2} \sum_{k \in \mathbb{Z}} \sum_{m=0}^{\infty} z^k (-1)^m q^{\frac{1}{2}m(m+1) + |k|(m+\frac{1}{2})}. \end{aligned}$$

The proposition follows by comparing the coefficients of z^k on both sides. \square

Theorem 2.7. *The q -dimension of the a_∞ -module of highest weight $\Lambda(\lambda)$ and level $-l$ is*

$$Q_{-l}^\lambda(q) = \sum_{\sigma \in S_l} (-1)^{\ell(\sigma)} Q_{-1}^{(k_1)}(q) \cdots Q_{-1}^{(k_l)}(q),$$

where $k_i = (\lambda + \rho - \sigma(\rho), \varepsilon_i)$.

Proof. Note that

$$\begin{aligned} \prod_{i=1}^l \frac{1}{(z_i q^{\frac{1}{2}})_\infty (z_i^{-1} q^{\frac{1}{2}})_\infty} &= \prod_{i=1}^l \left(\sum_{k \in \mathbb{Z}} z_i^k Q_{-1}^{(k)}(q) \right) \\ &= \sum_{\mathbf{k} \in \mathbb{Z}^l} z_1^{k_1} \cdots z_l^{k_l} Q_{-1}^{(k_1)}(q) \cdots Q_{-1}^{(k_l)}(q). \end{aligned}$$

Using Lemma 2.2, the character formula (2.7) and the Weyl denominator formula, we get

$$\begin{aligned} \sum_{\lambda} |z_j^{\lambda_i + l - i}| Q_{-l}^\lambda(q) &= |z_j^{l-i}| \sum_{\mathbf{k} \in \mathbb{Z}^l} z_1^{k_1} \cdots z_l^{k_l} Q_{-1}^{(k_1)}(q) \cdots Q_{-1}^{(k_l)}(q) \\ &= \sum_{\sigma \in S_l} (-1)^{\ell(\sigma)} \prod_{i=1}^l z_i^{(\sigma(\rho), \varepsilon_i)} \sum_{\mathbf{k} \in \mathbb{Z}^l} z_1^{k_1} \cdots z_l^{k_l} Q_{-1}^{(k_1)}(q) \cdots Q_{-1}^{(k_l)}(q). \end{aligned}$$

Comparing the coefficients of $\prod_{i=1}^l z_i^{\lambda_i + l - i}$ on both sides gives the result. \square

3. THE c_∞ -CORRELATION FUNCTIONS AND q -DIMENSION FORMULAS

3.1. Lie algebra c_∞ . Let

$$\bar{c}_\infty = \{(a_{ij})_{i,j \in \mathbb{Z}} \in \mathfrak{gl} \mid a_{ij} = -(-1)^{i+j} a_{1-j, 1-i}\}$$

be a Lie subalgebra of \mathfrak{gl} of type C [DJKM2]. Denote by c_∞ the central extension of \bar{c}_∞ given by the restriction of the 2-cocycle (2.4) to \bar{c}_∞ . Then c_∞ inherits from a_∞ a triangular decomposition with Cartan subalgebra $(c_\infty)_0$. We let

$$\begin{aligned} H_i^c &= E_{ii} + E_{-i, -i} - E_{i+1, i+1} - E_{1-i, 1-i}, \quad i \in \mathbb{N}, \\ H_0^c &= E_{0,0} - E_{1,1} + C. \end{aligned}$$

Denote by $\Lambda_i^c \in (c_\infty)_0^*$ the i -th fundamental weight of c_∞ , i.e. $\Lambda_i^c(H_j^c) = \delta_{ij}$.

The Dynkin diagram of c_∞ is:

$$\begin{array}{ccccccc} \circ & \rightrightarrows & \circ & \text{---} & \circ & \text{---} & \circ \cdots \\ 0 & & 1 & & 2 & & 3 \end{array}$$

The Lie algebra c_∞ is generated by

$$E_{ij} - (-1)^{i+j} E_{1-j, 1-i}, \quad i, j \in \mathbb{Z},$$

which can be organized into a generating series as

$$E(z, w) := \sum_{i,j} (E_{ij} - (-1)^{i+j} E_{1-j, 1-i}) z^{i-1} w^{-j}.$$

Following [TW], we introduce the following operators in c_∞ :

$$\begin{aligned} :\mathcal{C}(t): &= \sum_{r \in \frac{1}{2} + \mathbb{Z}_+} (t^r - t^{-r}) \left(E_{r+\frac{1}{2}, r+\frac{1}{2}} - E_{\frac{1}{2}-r, \frac{1}{2}-r} \right), \\ \mathcal{C}(t) &= :\mathcal{C}(t): + \frac{2}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} C. \end{aligned}$$

3.2. The 1-point c_∞ -function of level $-\frac{1}{2}$. Let $\chi(z) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} \chi_r z^{-r-\frac{1}{2}}$ be a free bosonic field with commutation relations

$$[\chi_r, \chi_s] = (-1)^{r+\frac{1}{2}} \delta_{r+s, 0}, \quad r, s \in \frac{1}{2} + \mathbb{Z}.$$

The Fock space $\mathfrak{F}^{-\frac{1}{2}}$ (cf. [FF, W1]) generated by $\chi(z)$ is a c_∞ -module of level $-\frac{1}{2}$ given by

$$E(z, w) = :\chi(z)\chi(-w):,$$

or equivalently by letting

$$E_{ij} - (-1)^{i+j} E_{1-j, 1-i} = (-1)^{-j} :\chi_{-i+\frac{1}{2}} \chi_{j-\frac{1}{2}}:.$$

When acting on $\mathfrak{F}^{-\frac{1}{2}}$ we have

$$\begin{aligned}
\mathbb{C}(t) &= \sum_{r \in \frac{1}{2} + \mathbb{Z}_+} \left((-1)^{r+\frac{1}{2}} t^r \chi_{-r} \chi_r - (-1)^{r+\frac{1}{2}} t^{-r} \chi_r \chi_{-r} \right) \\
&= \sum_{r \in \frac{1}{2} + \mathbb{Z}} (-1)^{r+\frac{1}{2}} t^r \chi_{-r} \chi_r, \\
:\mathbb{C}(t): &= \sum_{r \in \frac{1}{2} + \mathbb{Z}_+} (-1)^{r+\frac{1}{2}} (t^r - t^{-r}) : \chi_{-r} \chi_r : \\
&= \sum_{r \in \frac{1}{2} + \mathbb{Z}_+} \left((-1)^{r+\frac{1}{2}} t^r \chi_{-r} \chi_r - (-1)^{r+\frac{1}{2}} t^{-r} \chi_r \chi_{-r} \right).
\end{aligned}$$

Lemma 3.1. *We have $[\mathbb{C}(t), \chi_s] = -(t^s - t^{-s})\chi_s$.*

Proof. For $s > 0$ we compute that

$$\begin{aligned}
[\mathbb{C}(t), \chi_s] &= \sum_{r > 0} (-1)^{r+\frac{1}{2}} t^r [\chi_{-r}, \chi_s] \chi_r - (-1)^{r+\frac{1}{2}} t^{-r} \chi_r [\chi_{-r}, \chi_s] \\
&= -(t^s - t^{-s})\chi_s.
\end{aligned}$$

The case for $s < 0$ is similar. □

Let L_0 be the zero mode of the Virasoro field so that $[L_0, \chi_{-r}] = r\chi_{-r}$ and $L_0|0\rangle = 0$. Define the n -point c_∞ -function of level $-\frac{1}{2}$ as

$$\mathfrak{C}_{-\frac{1}{2}}^{(0)}(q; t_1, \dots, t_n) = \text{tr}_{\mathfrak{F}^{-\frac{1}{2}}} q^{L_0} \mathbb{C}(t_1) \cdots \mathbb{C}(t_n).$$

Theorem 3.1. *The 1-point c_∞ -function $\mathfrak{C}_{-\frac{1}{2}}^{(0)}(q; t)$ is equal to*

$$\begin{aligned}
&\frac{1}{(q^{\frac{1}{2}})_\infty (t^{-\frac{1}{2}} - t^{\frac{1}{2}})} - \frac{1}{(q^{\frac{1}{2}})_\infty (1 - q^{-\frac{1}{2}})} \frac{t^{\frac{1}{2}} (tq^{\frac{3}{2}})_\infty}{(tq)_\infty} {}_2\Phi_2(q^{\frac{1}{2}}, q^{\frac{1}{2}}; q^{\frac{3}{2}}, tq^{\frac{3}{2}}; q^2 t) \\
&+ \frac{1}{(q^{\frac{1}{2}})_\infty (1 - q^{-\frac{1}{2}})} \frac{t^{-\frac{1}{2}} (t^{-1}q^{\frac{3}{2}})_\infty}{(t^{-1}q)_\infty} {}_2\Phi_2(q^{\frac{1}{2}}, q^{\frac{1}{2}}; q^{\frac{3}{2}}, t^{-1}q^{\frac{3}{2}}; q^2 t^{-1}).
\end{aligned}$$

Proof. The Fock space $\mathfrak{F}^{-\frac{1}{2}}$ has a basis given by

$$v_\lambda = \chi_{-\lambda_1 + \frac{1}{2}} \chi_{-\lambda_2 + \frac{1}{2}} \cdots |0\rangle,$$

where $\lambda = (\lambda_1, \lambda_2, \dots)$ runs over all partitions. By Lemma 3.1

$$\mathbb{C}(t)v_\lambda = \left(\sum_{i=1}^l (t^{\lambda_i - \frac{1}{2}} - t^{-\lambda_i + \frac{1}{2}}) + \frac{1}{t^{-\frac{1}{2}} - t^{\frac{1}{2}}} \right) v_\lambda,$$

and hence we have

$$\mathfrak{C}_{-\frac{1}{2}}^{(0)}(q; t) = \frac{1}{(q^{\frac{1}{2}})_{\infty}(t^{-\frac{1}{2}} - t^{\frac{1}{2}})} + \sum_{l=1}^{\infty} q^{-\frac{l}{2}} \sum_{l(\lambda)=l} q^{|\lambda|} \sum_{i=1}^l (t^{\lambda_i - \frac{1}{2}} - t^{-\lambda_i + \frac{1}{2}}).$$

Now the theorem follows by applying (2.6) (with $x = 1$) twice. \square

We remark that the above q -hypergeometric series is again of type II.

3.3. A q -difference equation for c_{∞} -functions of level $-\frac{1}{2}$.

Theorem 3.2. *The n -point c_{∞} -function satisfies the q -difference equation:*

$$\mathfrak{C}_{-\frac{1}{2}}^{(0)}(q; qt_1, t_2, \dots, t_n) = \sum_{s=0}^{n-1} \sum_{1 < i_1 < \dots < i_s \leq n} \sum_{\epsilon_{i_a} = \pm 1} (-1)^{s + \#\epsilon} \mathfrak{C}_{-\frac{1}{2}}^{(0)}(q; t_1 t_{i_1}^{\epsilon_{i_1}} \dots t_{i_s}^{\epsilon_{i_s}}, \dots, \widehat{t}_{i_1}, \dots, \widehat{t}_{i_s}, \dots),$$

where $\#\epsilon$ stands for the number of ϵ_{i_a} 's that are equal to -1 .

Proof. For $r > 0$, by Lemma 3.1 we have

$$(3.1) \quad \mathbb{C}(t_2) \cdots \mathbb{C}(t_n) \chi_r = \sum_{S \subset \{2, \dots, n\}} (-1)^{|S|} \prod_{i \in S} (t_i^r - t_i^{-r}) \chi_r \prod_{i \notin S} \mathbb{C}(t_i),$$

and

$$(3.2) \quad \mathbb{C}(t_2) \cdots \mathbb{C}(t_n) \chi_{-r} = \sum_{S \subset \{2, \dots, n\}} (-1)^{|S|} \prod_{i \in S} (t_i^{-r} - t_i^r) \chi_{-r} \prod_{i \notin S} \mathbb{C}(t_i).$$

We will write tr instead of $\text{tr}_{\mathfrak{g}^{-\frac{1}{2}}}$ below. Let us simplify notation by setting

$$S_r^{\pm} := \text{tr } q^{L_0} \chi_{\mp r} \mathbb{C}(t_2) \cdots \mathbb{C}(t_n) \chi_{\pm r}.$$

Then (3.1) and (3.2) imply the following:

$$(3.3) \quad S_r^+ = \sum_{S \subset \{2, \dots, n\}} (-1)^{|S|} \prod_{i \in S} (t_i^r - t_i^{-r}) \text{tr } q^{L_0} \chi_{-r} \chi_r \prod_{i \notin S} \mathbb{C}(t_i),$$

and

$$(3.4) \quad S_r^- = \sum_{S \subset \{2, \dots, n\}} (-1)^{|S|} \prod_{i \in S} (t_i^{-r} - t_i^r) \text{tr } q^{L_0} \chi_r \chi_{-r} \prod_{i \notin S} \mathbb{C}(t_i).$$

It is clear that

$$\prod_{i \in S} (t_i^{\pm r} - t_i^{\mp r}) = \sum_{\epsilon_i = \pm 1, i \in S} (-1)^{\#\epsilon} \prod_{i \in S} (t_i^{\epsilon_i})^{\pm r}.$$

We will compute

$$\sum_{r > 0} \left((-1)^{r + \frac{1}{2}} t_1^r S_r^+ - (-1)^{r + \frac{1}{2}} t_1^{-r} S_r^- \right)$$

in two different ways. Using (3.3) and (3.4) it is equal to

$$\sum_{r>0} \sum_{S \subset \{2, \dots, n\}} (-1)^{|S|} \sum_{\epsilon_i = \pm 1, i \in S} (-1)^{\# \epsilon} \operatorname{tr} q^{L_0} \\ \times \left((-1)^{r+\frac{1}{2}} (t_1 \prod_{i \in S} (t_i^{\epsilon_i}))^r \chi_{-r} \chi_r - (-1)^{r+\frac{1}{2}} (t_1 \prod_{i \in S} (t_i^{\epsilon_i}))^{-r} \chi_r \chi_{-r} \right) \prod_{i \notin S} \mathbb{C}(t_i),$$

which, using the definition of $\mathbb{C}(t)$, is

$$(3.5) \quad \sum_{s=0}^{n-1} (-1)^s \sum_{1 < i_1 < \dots < i_s \leq n} \sum_{\epsilon_{i_a} = \pm 1} (-1)^{\# \epsilon} \mathfrak{C}_{-\frac{1}{2}}^{(0)}(t_1 t_{i_1}^{\epsilon_{i_1}} \dots t_{i_s}^{\epsilon_{i_s}}, \dots, \widehat{t}_{i_1}, \dots, \widehat{t}_{i_s}, \dots).$$

On the other hand, using the commutation relation $[L_0, \chi_{-r}] = r \chi_{-r}$, we see that for any r ,

$$\operatorname{tr} q^{L_0} \chi_{-r} \mathbb{C}(t_2) \dots \mathbb{C}(t_n) \chi_r = \operatorname{tr} \chi_r q^{L_0} \chi_{-r} \mathbb{C}(t_2) \dots \mathbb{C}(t_n) \\ = q^r \operatorname{tr} q^{L_0} \chi_r \chi_{-r} \mathbb{C}(t_2) \dots \mathbb{C}(t_n).$$

Now we have that

$$(3.6) \quad \sum_{r>0} \left((-1)^{r+\frac{1}{2}} t_1^r S_r^+ - (-1)^{r+\frac{1}{2}} t_1^{-r} S_r^- \right) = \operatorname{tr} q^{L_0} \tilde{\mathbb{C}}(qt) \mathbb{C}(t_2) \dots \mathbb{C}(t_n),$$

where

$$\tilde{\mathbb{C}}(t) = \sum_{r>0} \left((-1)^{r+\frac{1}{2}} t^r \chi_r \chi_{-r} - (-1)^{r+\frac{1}{2}} t^{-r} \chi_{-r} \chi_r \right).$$

One verifies that $\tilde{\mathbb{C}}(t) = \mathbb{C}(t)$, by observing that $\sum_{r>0} t^r = \frac{t^{\frac{1}{2}}}{1-t}$ coincides with $-\sum_{r>0} t^{-r} = \frac{t^{-\frac{1}{2}}}{t^{-1}-1}$. Equating (3.5) and (3.6) now completes the proof. \square

3.4. The n -point c_∞ -functions of level $l - \frac{1}{2}$. Let $l \in \mathbb{N}$. Let \mathfrak{F}^l be the Fock space generated by l pairs of free fermions $\psi^{\pm, p}(z) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} \psi_r^{\pm, p} z^{-r-\frac{1}{2}}$, $p = 1, \dots, l$, with non-trivial commutation relations

$$[\psi_r^{+, p}, \psi_s^{-, q}]_+ = \delta_{p, q} \delta_{r+s, 0}, \quad \text{for } r, s \in \frac{1}{2} + \mathbb{Z}.$$

Let $\mathfrak{F}^{l-\frac{1}{2}} \equiv \mathfrak{F}^{-\frac{1}{2}} \otimes \mathfrak{F}^l$. The Lie algebra c_∞ acts on $\mathfrak{F}^{l-\frac{1}{2}}$ by (see [W1])

$$\sum_{i, j \in \mathbb{Z}} (E_{i, j} - (-1)^{i+j} E_{1-j, 1-i}) z^{i-1} w^{-j} \\ = \sum_{k=1}^l (:\psi^{+, k}(z) \psi^{-, k}(w): + :\psi^{+, k}(-w) \psi^{-, k}(-z):) + :\chi(z) \chi(-w):.$$

It follows that the operator $C(t)$ acts on $\mathfrak{F}^{l-\frac{1}{2}}$ as

$$C(t) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} \sum_{p=1}^l t^r (\psi_{-r}^{+,p} \psi_r^{-,p} + \psi_{-r}^{-,p} \psi_r^{+,p}) + \sum_{r \in \frac{1}{2} + \mathbb{Z}} (-1)^{r+\frac{1}{2}} t^r \chi_{-r} \chi_r.$$

From [W1, Lemma 4.6] we have an action of the Lie superalgebra $\mathfrak{osp}(1, 2l)$ on $\mathfrak{F}^{l-\frac{1}{2}}$ that commutes with the action of c_∞ . It is known that the finite-dimensional simple $\mathfrak{osp}(1, 2l)$ -modules, denoted by $V_\lambda(\mathfrak{osp}(1, 2l))$, are parameterized by partitions λ of length $\leq l$ as highest weights. Associated to a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ we define a highest weight $\Lambda(\lambda)$ of c_∞ via

$$\Lambda(\lambda) = (l - \frac{1}{2} - j) \Lambda_0^c + \sum_{k=1}^j \Lambda_{\lambda_k}^c,$$

if $\lambda_1 \geq \dots \geq \lambda_j > \lambda_{j+1} = \dots = \lambda_l = 0$.

Proposition 3.1. [W1, Theorem 4.3] *We have the $(\mathfrak{osp}(1, 2l), c_\infty)$ -module decomposition*

$$\mathfrak{F}^{l-\frac{1}{2}} \cong \bigoplus_{\lambda} V_\lambda(\mathfrak{osp}(1, 2l)) \otimes L(c_\infty; \Lambda(\lambda), l - \frac{1}{2}).$$

Let $\mathbf{t} = (t_1, \dots, t_n)$. The Bloch-Okounkov n -point c_∞ -function of level $l - \frac{1}{2}$ (associated to a partition λ of length $\leq l$) is defined as

$$\mathfrak{C}_{l-\frac{1}{2}}^\lambda(q; \mathbf{t}) \equiv \mathfrak{C}_{l-\frac{1}{2}}^\lambda(q; t_1, \dots, t_n) = \text{tr}_{L(c_\infty; \Lambda(\lambda), -l)} q^{L_0} C(t_1) C(t_2) \cdots C(t_n).$$

Lemma 3.2. *The character $\text{ch}_\lambda^{\mathfrak{osp}}(z_1, \dots, z_l) := \text{tr}_{V_\lambda(\mathfrak{osp}(1, 2l))}(z_1^{e_{11}} \cdots z_l^{e_{ll}})$ of the irreducible $\mathfrak{osp}(1, 2l)$ -module of highest weight λ is given by*

$$\text{ch}_\lambda^{\mathfrak{osp}}(z_1, \dots, z_l) = \frac{\left| z_j^{\lambda_i + l - i + \frac{1}{2}} - z_j^{-(\lambda_i + l - i + \frac{1}{2})} \right|}{\left| z_j^{l - i + \frac{1}{2}} - z_j^{-(l - i + \frac{1}{2})} \right|}.$$

Proof. Recall (cf. [K2]) that the positive even roots for $\mathfrak{osp}(1, 2l)$ are given by $\Phi_0 = \{2\varepsilon_i, \varepsilon_i \pm \varepsilon_j | 1 \leq i \neq j \leq l\}$ and the positive odd roots by $\Phi_1 = \{\varepsilon_i | 1 \leq i \leq l\}$. The character formula for $V_\lambda(\mathfrak{osp}(1, 2l))$ is given by [K2]

$$\text{ch} V_\lambda(\mathfrak{osp}(1, 2l)) = \frac{\prod_{\alpha \in \Phi_1} (1 + e^{-\alpha}) \sum_{\sigma \in W(C_l)} (-1)^{\ell(\sigma)} e^{\sigma(\lambda + \rho_{\mathfrak{osp}}) - \rho_{\mathfrak{osp}}}}{\prod_{\alpha \in \Phi_0} (1 - e^{-\alpha})},$$

where $\rho_{\mathfrak{osp}} = \frac{1}{2} \sum_{\alpha \in \Phi_0} \alpha - \frac{1}{2} \sum_{\alpha \in \Phi_1} \alpha$.

Denote by Φ_B the root system for type B , i.e. $\Phi_B = \{\varepsilon_i, \varepsilon_i \pm \varepsilon_j | i \neq j\}$. It is easy to see that ρ_{osp} is equal to ρ_B . Note that

$$\frac{1 + e^{-\varepsilon_i}}{1 - e^{-2\varepsilon_i}} = \frac{1}{1 - e^{-\varepsilon_i}}.$$

Since the Weyl group is the same for type C and type B , the character of $V_\lambda(\mathfrak{osp}(1, 2l))$ coincides with irreducible $\mathfrak{so}(2l+1)$ -character of highest weight λ , which is known (cf. [FH]) to be given by the right-hand side of the formula in the lemma. \square

Let us denote by $\mathbf{F}(z, q, \mathbf{t}) = \text{tr}_{\mathfrak{F}^1} z^{\varepsilon_{11}} q^{L_0} \mathbf{C}(t_1) \cdots \mathbf{C}(t_n)$.

Lemma 3.3. *We have the following q -series identity:*

$$\text{tr}_{\mathfrak{F}^{-\frac{1}{2}}} q^{L_0} \mathbf{C}(t_1) \cdots \mathbf{C}(t_n) \cdot \prod_{i=1}^l \mathbf{F}(z_i, q, \mathbf{t}) = \sum_{\lambda} \text{ch}_{\lambda}^{osp}(z_1, \dots, z_l) \mathfrak{C}_{l-\frac{1}{2}}^{\lambda}(q; \mathbf{t}).$$

Proof. The identity results by applying $\text{tr}_{\mathfrak{F}^{-\frac{1}{2}}} z_1^{\varepsilon_{11}} \cdots z_l^{\varepsilon_{ll}} q^{L_0} \mathbf{C}(t_1) \cdots \mathbf{C}(t_n)$ to both sides of Proposition 3.1. On the left-hand side, $z_i^{\varepsilon_{ii}}$ only acts on the i^{th} tensor factor of \mathfrak{F}^l and not on $\mathfrak{F}^{-\frac{1}{2}}$. For the right-hand side, $z_1^{\varepsilon_{11}} \cdots z_l^{\varepsilon_{ll}}$ acts only on the first tensor factor of $V_\lambda(\mathfrak{osp}(1, 2l))$ and $q^{L_0} \mathbf{C}(t_1) \cdots \mathbf{C}(t_n)$ acts only on the second tensor factor. \square

Define the theta function

$$\Theta(t) := (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(q)_{\infty}^{-2} (qt)_{\infty} (qt^{-1})_{\infty},$$

and let $\Theta^{(k)}(t) = (t \frac{d}{dt})^k \Theta(t)$ for $k \in \mathbb{Z}_+$. Denote

$$F_{bo}(q; \mathbf{t}) := \frac{1}{(q)_{\infty}} \sum_{\sigma \in S_n} \frac{\det \left(\frac{\Theta^{(j-i+1)}(t_{\sigma(1)} \cdots t_{\sigma(n-j)})}{(j-i+1)!} \right)_{i,j=1}^n}{\Theta(t_{\sigma(1)}) \Theta(t_{\sigma(1)} t_{\sigma(2)}) \cdots \Theta(t_{\sigma(1)} t_{\sigma(2)} \cdots t_{\sigma(n)})}$$

where it is understood below that $\frac{1}{(-k)!} = 0$, for $k \in \mathbb{N}$. We recall the formula for the original Bloch-Okounkov n -point correlation function of the a_{∞} -module $\mathfrak{F}_{(k)}^1$ of level 1 [BO, Ok] is given by

$$(3.7) \quad \text{tr}_{\mathfrak{F}_{(k)}^1} (q^{L_0} \mathbf{A}(t_1) \cdots \mathbf{A}(t_n)) = q^{\frac{k^2}{2}} (t_1 \cdots t_n)^k F_{bo}(q; t_1, \dots, t_n).$$

Theorem 3.3. *The n -point c_{∞} -function, $\mathfrak{C}_{l-\frac{1}{2}}^{\lambda}(q; t_1, \dots, t_n)$, is equal to*

$$\mathfrak{C}_{-\frac{1}{2}}^{(0)}(q; \mathbf{t}) \cdot \sum_{\sigma \in W(B_l)} (-1)^{\ell(\sigma)} q^{\frac{\|\lambda + \rho_B - \sigma(\rho_B)\|^2}{2}} \prod_{a=1}^l \left(\sum_{\vec{\epsilon}_a \in \{\pm 1\}^n} [\vec{\epsilon}_a] (\Pi \mathbf{t}^{\vec{\epsilon}_a})^{k_a} F_{bo}(q; \mathbf{t}^{\vec{\epsilon}_a}) \right),$$

where $k_a = (\lambda + \rho_B - \sigma(\rho_B), \varepsilon_a)$, and for an expression $\vec{\epsilon} = (\epsilon_1, \dots, \epsilon_n)$ we define $[\vec{\epsilon}] = \epsilon_1 \cdots \epsilon_n$ and $\Pi \mathbf{t}^{\vec{\epsilon}} = t_1^{\epsilon_1} \cdots t_n^{\epsilon_n}$.

Proof. This proof mirrors the one used in [TW]. The Weyl denominator of type B_l (also the denominator of $\mathbf{ch}_\lambda^{\text{osp}}$) reads that

$$|z_j^{l-i+\frac{1}{2}} - z_j^{-(l-i+\frac{1}{2})}| = \sum_{\sigma \in W(B_l)} (-1)^{\ell(\sigma)} \mathbf{z}^{\sigma(\rho_B)},$$

where we denote $\mathbf{z}^\mu = z_1^{\mu_1} \cdots z_l^{\mu_l}$, for $\mu = (\mu_1, \dots, \mu_l)$. It follows from Lemma 3.3 and $\mathbf{C}(t) = \mathbf{A}(t) - \mathbf{A}(t^{-1})$ that

$$\begin{aligned} & \sum_{\sigma \in W(B_l)} (-1)^{\ell(\sigma)} \mathbf{z}^{\sigma(\rho_B)} \cdot \mathfrak{C}_{-\frac{1}{2}}^{(0)}(q; \mathbf{t}) \\ & \quad \times \prod_{a=1}^l \left(\sum_{k_a \in \mathbb{Z}} z_a^{k_a} q^{\frac{k_a^2}{2}} \sum_{\vec{\epsilon}_a \in \{\pm 1\}^n} [\vec{\epsilon}_a] \cdot (\Pi \mathbf{t}^{\vec{\epsilon}_a})^{k_a} F_{bo}(q; \mathbf{t}^{\vec{\epsilon}_a}) \right) \\ & = \sum_{\lambda} |z_j^{\lambda_i+l-i+\frac{1}{2}} - z_j^{-(\lambda_i+l-i+\frac{1}{2})}| \cdot \mathfrak{C}_{l-\frac{1}{2}}^\lambda(q; \mathbf{t}), \end{aligned}$$

where we have used (3.7) in the above calculation. Among the monomials \mathbf{z}^μ in the expansion of $|z_j^{\lambda_i+l-i+\frac{1}{2}} - z_j^{-(\lambda_i+l-i+\frac{1}{2})}|$, there is exactly one dominant monomial with $\mu_1 \geq \dots \geq \mu_l \geq 0$, which is $\mathbf{z}^{\lambda+\rho_B}$. The theorem follows by comparing the coefficient of $\mathbf{z}^{\lambda+\rho_B}$ on both sides of the above equation. \square

3.5. The q -dimension of a c_∞ -module of level $l - \frac{1}{2}$. Let $l \in \mathbb{N}$. The q -dimension of the c_∞ -module $L(c_\infty; \Lambda(\lambda), l - \frac{1}{2})$ is

$${}^c\mathbf{Q}_{l-\frac{1}{2}}^\lambda(q) := \text{tr}_{L(c_\infty; \Lambda(\lambda), l-\frac{1}{2})} q^{L_0}.$$

We can derive the following q -dimension formula from the $(\mathfrak{osp}(1, 2l), c_\infty)$ -duality (see Proposition 3.1), where the second formula is easily seen to be equivalent to the first by [TW, Lemma 3.10].

Theorem 3.4. *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition of length $\leq l$. We have*

$$\begin{aligned} {}^c\mathbf{Q}_{l-\frac{1}{2}}^\lambda(q) &= \frac{1}{(q^{\frac{1}{2}})_\infty(q)^l} \cdot \sum_{\sigma \in W(B_l)} (-1)^{\ell(\sigma)} q^{\frac{\|\lambda+\rho_B-\sigma(\rho_B)\|^2}{2}} \\ &= \frac{1}{(q^{\frac{1}{2}})_\infty(q)^l} \cdot q^{\frac{\|\lambda\|^2}{2}} \prod_{1 \leq i \leq l} \left(1 - q^{\lambda_i+l-i+\frac{1}{2}}\right) \times \\ & \quad \times \prod_{1 \leq i < j \leq l} \left(1 - q^{\lambda_i-\lambda_j+j-i}\right) \left(1 - q^{\lambda_i+\lambda_j+2l-i-j+1}\right). \end{aligned}$$

Proof. By applying $\text{tr } z_1^{e_{11}} \cdots z_l^{e_{ll}} q^{L_0}$ to both sides of the duality in Proposition 3.1, we obtain

$$\prod_{i=1}^l (\text{tr } \mathfrak{F}^1 z_i^{e_{ii}} q^{L_0}) \text{tr } \mathfrak{F}^{-\frac{1}{2}} q^{L_0} = \sum_{\lambda} \text{ch}_{\lambda}^{osp}(z_1, \dots, z_l)^c \mathbf{Q}_{\lambda}^{l-\frac{1}{2}}(q).$$

Noting by the Jacobi triple product identity that

$$\text{tr } \mathfrak{F}^1 z^{e_{ii}} q^{L_0} = \prod_{r \geq 0} (1 + q^{r+\frac{1}{2}} z)(1 + q^{r+\frac{1}{2}} z^{-1}) = \frac{1}{(q)_{\infty}} \sum_{k \in \mathbb{Z}} z^k q^{\frac{k^2}{2}}$$

and that

$$\text{tr } \mathfrak{F}^{-\frac{1}{2}} q^{L_0} = \frac{1}{(q^{\frac{1}{2}})_{\infty}},$$

a completely analogous argument as for Theorem 3.3 applies. \square

3.6. The n -point c_{∞} -functions of level $-l$. Let $l \in \mathbb{N}$. Let \mathfrak{F}^{-l} denote the Fock space generated l pairs of free bosonic fields $\gamma^{\pm, p}(z) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} \gamma_r^{\pm, p} z^{-r-\frac{1}{2}}$ ($p = 1, \dots, l$) with non-trivial commutation relations

$$[\gamma_r^{+, p}, \gamma_s^{-, q}] = \delta_{pq} \delta_{r+s, 0}, \quad \text{for } r, s \in \frac{1}{2} + \mathbb{Z}.$$

According to [W1, Section 5.2] there is an action of c_{∞} on \mathfrak{F}^{-l} , from which we conclude that

$$\mathbb{C}(t) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \sum_{p=1}^l t^r (-\gamma_{-r}^{+, p} \gamma_r^{-, p} + \gamma_{-r}^{-, p} \gamma_r^{+, p}).$$

There is also an action of the Lie group $O(2l)$ on \mathfrak{F}^{-l} which commutes with the action of c_{∞} .

Denote the parameter set for simple $O(2l)$ -modules (cf. [BtD, W1]) by

$$\begin{aligned} \mathcal{C} := & \{(\lambda_1, \lambda_2, \dots, \lambda_l) \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0, \lambda_i \in \mathbb{Z}\} \\ & \cup \{(\lambda_1, \lambda_2, \dots, \lambda_{l-1}, 0) \otimes \det, (\lambda_1, \lambda_2, \dots, \lambda_{l-1}, 0) \mid \\ & \lambda_1 \geq \dots \geq \lambda_{l-1} \geq 0, \lambda_i \in \mathbb{Z}\}. \end{aligned}$$

Associated to $\lambda \in \mathcal{C}$ we define a highest weight $\Lambda(\lambda)$ for c_{∞}

$$\Lambda(\lambda) = (-l - \lambda_1) \Lambda_0^c + \sum_{k=1}^l (\lambda_k - \lambda_{k+1}) \Lambda_0^c,$$

if $\lambda = (\lambda_1, \dots, \lambda_l)$ with $\lambda_l > 0$,

$$\Lambda(\lambda) = (-l - \lambda_1) \Lambda_0^c + \sum_{k=1}^j (\lambda_k - \lambda_{k+1}) \Lambda_0^c,$$

if $\lambda = (\lambda_1, \dots, \lambda_j, 0, \dots, 0)$, and

$$\Lambda(\lambda) = (-l - \lambda_1)\Lambda_0^c + \sum_{k=1}^{j-1} (\lambda_k - \lambda_{k+1})\Lambda_0^c + (\lambda_j - 1)\Lambda_j^c + \Lambda_{2l-j}^c,$$

if $\lambda = (\lambda_1, \dots, \lambda_j, 0, \dots, 0) \otimes \det$.

Proposition 3.2. [W1, Theorem 5.3] *We have the following $(O(2l), c_\infty)$ -module decomposition:*

$$\mathfrak{F}^{-l} \cong \bigoplus_{\lambda \in \mathcal{C}} V_\lambda(O(2l)) \otimes L(c_\infty; \Lambda(\lambda), -l),$$

where $V_\lambda(O(2l))$ is the irreducible $O(2l)$ -module parameterized by λ .

Definition 3.1. The n -point c_∞ -function of level $-l$ (associated to a partition λ of length $\leq l$) is

$$\mathfrak{C}_{-l}^\lambda(q; t_1, \dots, t_n) = \begin{cases} \text{tr}_{L(c_\infty; \Lambda(\lambda), -l)} q^{L_0} \mathbb{C}(t_1) \cdots \mathbb{C}(t_n), & \lambda_l \neq 0, \\ \text{tr}_{L(c_\infty; \Lambda(\lambda), -l) \oplus L(c_\infty; \Lambda(\lambda \otimes \det), -l)} q^{L_0} \mathbb{C}(t_1) \cdots \mathbb{C}(t_n), & \lambda_l = 0. \end{cases}$$

Proposition 3.3. *The n -point function of level -1 , $\mathfrak{C}_{-1}^{(m)}(q; t_1, \dots, t_n)$, is given by*

$$[z^m] \sum_{\vec{\epsilon}_a \in \{\pm 1\}^n} [\vec{\epsilon}_a] \text{tr}_{\mathfrak{F}^{-1}} z^{e_{11}} q^{L_0} \mathbf{A}(t_1^{\epsilon_1}) \cdots \mathbf{A}(t_n^{\epsilon_n}).$$

Proof. Since by definition $\mathbb{C}(t) = \mathbf{A}(t) - \mathbf{A}(t^{-1})$, we have

$$\begin{aligned} \mathbb{C}(t_1) \cdots \mathbb{C}(t_n) &= \prod_{j=1}^n (\mathbf{A}(t_j) - \mathbf{A}(t_j^{-1})) \\ &= \sum_{\vec{\epsilon} \in \{\pm 1\}^n} \epsilon_1 \epsilon_2 \cdots \epsilon_n \mathbf{A}(t_1^{\epsilon_1}) \mathbf{A}(t_2^{\epsilon_2}) \cdots \mathbf{A}(t_n^{\epsilon_n}). \end{aligned}$$

Recall that

$$\mathfrak{F}^{-1} \cong \bigoplus_{m \in \mathbb{Z}} \mathfrak{F}_{(m)}^{-1}$$

where $\mathfrak{F}_{(m)}^{-1}$ is the m -eigenspace of e_{11} . Proposition 3.2 states that $\mathfrak{F}_{(m)}^{-1} \cong \mathfrak{F}_{(-m)}^{-1}$ as c_∞ -modules for $m \neq 0$, and also that $\mathfrak{F}_{(0)}^{-1} \cong L(c_\infty; \Lambda(\emptyset), -1) \oplus L(c_\infty; \Lambda(\emptyset \otimes \det), -1)$. Noting that

$$\begin{aligned} \text{tr}_{\mathfrak{F}^{-1}} z^{e_{11}} q^{L_0} \mathbb{C}(t_1) \cdots \mathbb{C}(t_n) &= \sum_{m \in \mathbb{Z}} z^m \text{tr}_{\mathfrak{F}_{(m)}^{-1}} q^{L_0} \mathbb{C}(t_1) \cdots \mathbb{C}(t_n) \\ &= \sum_{m \in \mathbb{Z}} z^m \mathfrak{C}_{-1}^{(|m|)}(q; t_1, \dots, t_n), \end{aligned}$$

the result follows. \square

Lemma 3.4. [TW, Lemma 3.2] Denote by $\text{ch}_\lambda^o(z_1, \dots, z_l)$ the character of the irreducible $O(2l)$ -module $V_\lambda(O(2l))$. Then

$$\text{ch}_\lambda^o(z_1, \dots, z_l) = c_\lambda \frac{\left| z_j^{\lambda_i+l-i} + z_j^{-(\lambda_i+l-i)} \right|}{\left| z_j^{l-i} + z_j^{-(l-i)} \right|},$$

where $c_\lambda = 1$ if $\lambda_l = 0$, and $c_\lambda = 2$ if $\lambda_l \neq 0$.

Theorem 3.5. Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition. The function $\mathfrak{C}_{-l}^\lambda(q; t_1, \dots, t_n)$ is equal to

$$\sum_{\sigma \in W(D_l)} (-1)^{\ell(\sigma)} \mathfrak{C}_{-1}^{(k_1)}(q; t_1, \dots, t_n) \cdots \mathfrak{C}_{-1}^{(k_l)}(q; t_1, \dots, t_n),$$

where $k_i = (\lambda + \rho - \sigma(\rho), \varepsilon_i) \geq 0$.

Proof. Apply the trace of $z_1^{e_{11}} \cdots z_l^{e_{ll}} q^{L_0} \mathbf{C}(t_1) \cdots \mathbf{C}(t_n)$ to both sides of the isomorphism in Proposition 3.2. On the left-hand side, $z_i^{e_{ii}}$ acts only on the i^{th} tensor factor, so it becomes

$$\begin{aligned} \prod_{i=1}^l \text{tr}_{\mathfrak{F}^{-1}} z_i^{e_{ii}} q^{L_0} \mathbf{C}(t_1) \cdots \mathbf{C}(t_n) &= \prod_{i=1}^l \sum_{m_i \in \mathbb{Z}} z_i^{m_i} \mathfrak{C}_{-1}^{(|m_i|)}(q; t_1, \dots, t_n) \\ &= \sum_{\mathbf{m} \in \mathbb{Z}^l} z_1^{m_1} \cdots z_l^{m_l} \mathfrak{C}_{-1}^{(|m_1|)}(q; t_1, \dots, t_n) \cdots \mathfrak{C}_{-1}^{(|m_l|)}(q; t_1, \dots, t_n). \end{aligned}$$

On the right-hand side, $z_1^{e_{11}} \cdots z_l^{e_{ll}}$ acts only on the module $V_\lambda(O(2l))$ while $q^{L_0} \mathbf{C}(t_1) \cdots \mathbf{C}(t_n)$ acts on $L(c_\infty; \Lambda(\lambda), -l)$, so it becomes

$$\begin{aligned} \sum_{\lambda \in \mathfrak{C}} \text{ch}_\lambda^o(z_1, \dots, z_l) \mathfrak{C}_{-l}^\lambda(q; t_1, \dots, t_n) \\ = \sum_{\lambda \in \mathfrak{C}} c_\lambda \frac{\left| z_j^{\lambda_i+l-i} + z_j^{-(\lambda_i+l-i)} \right|}{\left| z_j^{l-i} + z_j^{-(l-i)} \right|} \cdot \mathfrak{C}_{-l}^\lambda(q, t_1, \dots, t_n). \end{aligned}$$

Thus, equating both sides gives us

$$\begin{aligned} \sum_{\mathbf{m} \in \mathbb{Z}^l} z_1^{m_1} \cdots z_l^{m_l} \mathfrak{C}_{-1}^{(|m_1|)}(q; t_1, \dots, t_n) \cdots \mathfrak{C}_{-1}^{(|m_l|)}(q; t_1, \dots, t_n) \\ = \sum_{\lambda \in \mathfrak{C}} c_\lambda \frac{\left| z_j^{\lambda_i+l-i} + z_j^{-(\lambda_i+l-i)} \right|}{\left| z_j^{l-i} + z_j^{-(l-i)} \right|} \cdot \mathfrak{C}_{-l}^\lambda(q, t_1, \dots, t_n). \end{aligned}$$

Multiplying both sides by the Weyl denominator of type D_l , i.e.,

$$\frac{1}{2} |z_j^{l-i} + z_j^{-(l-i)}| = \sum_{\sigma \in W(D_l)} (-1)^{\ell(\sigma)} \mathbf{z}^{\sigma(\rho)},$$

we obtain that

$$\begin{aligned} \sum_{\sigma \in W(D_l)} (-1)^{\ell(\sigma)} \mathbf{z}^{\sigma(\rho)} \sum_{\mathbf{m} \in \mathbb{Z}^l} z_1^{m_1} \cdots z_l^{m_l} \mathfrak{C}_{-1}^{(|m_1|)}(q; t_1, \dots, t_n) \cdots \mathfrak{C}_{-1}^{(|m_l|)}(q; t_1, \dots, t_n) \\ = \sum_{\lambda \in \mathcal{C}} c_\lambda / 2 \cdot \left| z_j^{\lambda_i + l - i} + z_j^{-(\lambda_i + l - i)} \right| \mathfrak{C}_{-l}^\lambda(q; t_1, \dots, t_n). \end{aligned}$$

The result follows by comparing the coefficients of $\prod_{i=1}^l z_i^{\lambda_i + l - i}$ on both sides, noting that its coefficient in $|z_j^{\lambda_i + l - i} + z_j^{-(\lambda_i + l - i)}|$ is precisely $2/c_\lambda$. \square

The 1-point (respectively, 2-point) c_∞ -function of level $-l$ now follows by combining Proposition 3.3, Theorem 3.5, Remark 2.1 (respectively, Remark 2.2).

3.7. The q -dimension of a c_∞ -module of level $-l$. Let us denote by ${}^c\mathbf{Q}_{-l}^\lambda(q)$ the q -dimension of $L(c_\infty; \Lambda(\lambda), -l)$, or in the case that $\lambda_l = 0$, the q -dimension of $L(c_\infty; \Lambda(\lambda), -l) \oplus L(c_\infty; \Lambda(\lambda \otimes \det), -l)$.

Proposition 3.4. *We have the following q -dimension formula of level -1 (for $k \in \mathbb{Z}_+$):*

$${}^c\mathbf{Q}_{-1}^{(k)}(q) = \frac{1}{(q)_\infty^2} \sum_{m \geq 0} (-1)^m q^{\frac{1}{2}m(m+1) + |k|(m + \frac{1}{2})}.$$

Proof. We can identify $L(c_\infty; \Lambda^c(k), -1) = L(a_\infty; \Lambda^a(k), -1)$ in \mathfrak{F}^{-1} for $k > 0$ by comparing Propositions 2.2 and 3.2, where we have temporarily used the superscripts a, c to distinguish the weights for a_∞ and c_∞ respectively; also we have $L(c_\infty; \Lambda^c(0), -1) \oplus L(c_\infty; \Lambda^c((0) \otimes \det), -1) = L(a_\infty; \Lambda^a(0), -1)$. Now the result follows from Proposition 2.3. \square

Theorem 3.6. *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition. We have the following q -dimension formula:*

$${}^c\mathbf{Q}_{-l}^\lambda(q) = \sum_{\sigma \in W(D_l)} (-1)^{\ell(\sigma)} {}^c\mathbf{Q}_{-1}^{(k_1)}(q) \cdots {}^c\mathbf{Q}_{-1}^{(k_l)}(q),$$

where $k_i = (\lambda + \rho - \sigma(\rho), \varepsilon_i)$.

Proof. The proof is similar to that of Theorem 3.5, and we omit the details. \square

3.8. The n -point c_∞ -functions of level $-l - \frac{1}{2}$. Denote by $\mathcal{C}_{\frac{1}{2}}$ the following parameter set for irreducible $O(2l+1)$ -modules ([BtD, W1]):

$$\{(\lambda_1, \lambda_2, \dots, \lambda_l) \otimes \det, (\lambda_1, \lambda_2, \dots, \lambda_l) \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l \geq 0, \lambda_i \in \mathbb{Z}\}.$$

Associated to $\lambda \in \mathcal{C}_{\frac{1}{2}}$ we define the highest weight $\Lambda(\lambda)$ of c_∞ as

$$\Lambda(\lambda) = (-l - \lambda_1 - \frac{1}{2})\Lambda_0^c + \sum_{k=1}^j (\lambda_k - \lambda_{k+1})\Lambda_k^c,$$

if $\lambda = (\lambda_1, \dots, \lambda_j, 0, \dots, 0)$ and

$$\Lambda(\lambda) = (-l - \lambda_1 - \frac{1}{2})\Lambda_0^c + \sum_{k=1}^{j-1} (\lambda_k - \lambda_{k+1})\Lambda_k^c + (\lambda_j - 1)\Lambda_j^c + \Lambda_{2l-j+1}^c,$$

if $\lambda = (\lambda_1, \dots, \lambda_j, 0, \dots, 0) \otimes \det$.

According to [W1, Section 6.2], there exist commuting actions of c_∞ and of $O(2l+1)$ on $\mathfrak{F}^{-l-\frac{1}{2}}$.

Proposition 3.5. [W1, Theorem 6.2] *We have the following $(O(2l+1), c_\infty)$ -module decomposition:*

$$\mathfrak{F}^{-l-\frac{1}{2}} \cong \bigoplus_{\lambda \in \mathcal{C}_{\frac{1}{2}}} V_\lambda(O(2l+1)) \otimes L(c_\infty; \Lambda(\lambda), -l - \frac{1}{2}),$$

where $V_\lambda(O(2l+1))$ is the irreducible $O(2l+1)$ -module parameterized by λ .

The operator $C(t)$ acting on $\mathfrak{F}^{-l-\frac{1}{2}}$ can be expressed now as

$$C(t) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \sum_{p=1}^l t^r (-\gamma_{-r}^{+,p} \gamma_r^{-,p} + \gamma_{-r}^{-,p} \gamma_r^{+,p}) + \sum_{r \in \mathbb{Z} + \frac{1}{2}} (-1)^{r+\frac{1}{2}} t^r \chi_{-r} \chi_r.$$

Definition 3.2. The Bloch-Okounkov n -point c_∞ -function of level $-l - \frac{1}{2}$ (associated to a partition λ of length $\leq l$) is defined as

$$\mathfrak{C}_{-l-\frac{1}{2}}^\lambda(q; t_1, \dots, t_n) = \text{tr}_{L(c_\infty; \Lambda(\lambda), -l-\frac{1}{2}) \oplus L(c_\infty; \Lambda(\lambda \otimes \det), -l-\frac{1}{2})} q^{L_0} C(t_1) \cdots C(t_n).$$

Theorem 3.7. *The function $\mathfrak{C}_{-l-\frac{1}{2}}^\lambda(q; t_1, \dots, t_n)$ is given by*

$$\mathfrak{C}_{-\frac{1}{2}}^{(0)}(q; t_1, \dots, t_n) \cdot \sum_{\sigma \in W(B_l)} (-1)^{\ell(\sigma)} \mathfrak{C}_{-1}^{(k_1)}(q; t_1, \dots, t_n) \cdots \mathfrak{C}_{-1}^{(k_l)}(q; t_1, \dots, t_n),$$

where $k_i = (\lambda + \rho_B - \sigma(\rho_B), \epsilon_i) \geq 0$.

Proof. The proof is similar to that of Theorem 3.5, where instead we use the Weyl group of type B_l and replace the character ch_λ^o therein with

$$\text{ch}_\lambda^b(z_1, \dots, z_l) = \frac{\left| z_j^{\lambda_i + l - i + \frac{1}{2}} - z_j^{-(\lambda_i + l - i + \frac{1}{2})} \right|}{\left| z_j^{l - i + \frac{1}{2}} - z_j^{-(l - i + \frac{1}{2})} \right|}.$$

Note that the factor c_λ therein does not appear in this computation, and also note that $z_i^{e_{ii}}$ does not act on $\mathfrak{F}^{-\frac{1}{2}}$ which produces the appearance of the factor $\mathfrak{C}_{-\frac{1}{2}}^{(0)}(q; t_1, \dots, t_n)$ in the formula. Finally $V_\lambda(O(2l+1))$ and $V_{\lambda \otimes \det}(O(2l+1))$ are isomorphic as modules over the Lie algebra of $O(2l+1)$, and hence they have the same character. \square

The 1-point c_∞ -function of level $-l$ now follows by combining Theorem 3.1, Proposition 3.3, Theorem 3.7, Remark 2.1.

3.9. The q -dimension of a c_∞ -module of level $-l - \frac{1}{2}$. Again, let us denote by ${}^c\mathbf{Q}_{-l-\frac{1}{2}}^\lambda(q)$ the q -dimension of $L(c_\infty; \Lambda(\lambda), -l - \frac{1}{2}) \oplus L(c_\infty; \Lambda(\lambda \otimes \det), -l - \frac{1}{2})$.

Theorem 3.8. *The q -dimension ${}^c\mathbf{Q}_{-l-\frac{1}{2}}^\lambda$ is*

$${}^c\mathbf{Q}_{-l}^\lambda(q) = \frac{1}{(q^{\frac{1}{2}})_\infty} \sum_{\sigma \in W(B_l)} (-1)^{\ell(\sigma)} {}^c\mathbf{Q}_{-1}^{k_1}(q) \cdots {}^c\mathbf{Q}_{-1}^{k_l}(q),$$

where $k_i = (\lambda + \rho_B - \sigma(\rho_B), \varepsilon_i)$.

Proof. This can be proved similarly to Theorem 3.7 and we will skip the details. \square

4. THE d_∞ -CORRELATION FUNCTIONS AND q -DIMENSION FORMULAS

As the methods involved in the d_∞ case are similar to the cases of a_∞ and c_∞ treated in the previous sections, we shall be more sketchy in this section.

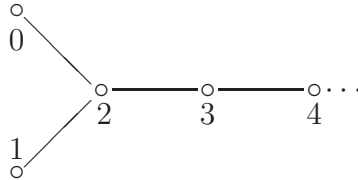
4.1. Lie algebra d_∞ . Let

$$\bar{d}_\infty = \{(a_{ij})_{i,j \in \mathbb{Z}} \in \mathfrak{gl} \mid a_{ij} = -a_{1-j, 1-i}\}$$

be a Lie subalgebra of \mathfrak{gl} of type D [DJKM2]. Denote by $d_\infty = \bar{d}_\infty \oplus \mathbb{C}C$ the central extension given by the restriction of 2-cocycle (2.4) to \bar{d}_∞ . Then d_∞ has a natural triangular decomposition induced from a_∞ with Cartan subalgebra $(d_\infty)_0 = (a_\infty)_0 \cap d_\infty$. Given $\Lambda \in (d_\infty)_0^*$, we let

$$\begin{aligned} H_i^d &= E_{ii} + E_{-i,-i} - E_{i+1,i+1} - E_{-i+1,-i+1} \quad (i \in \mathbb{N}), \\ H_0^d &= E_{0,0} + E_{-1,-1} - E_{2,2} - E_{1,1} + 2C. \end{aligned}$$

Denote by Λ_i^d the i -th fundamental weight of d_∞ , i.e. $\Lambda_i^d(H_j^d) = \delta_{ij}$. The Dynkin diagram of d_∞ is:



4.2. The n -point d_∞ -functions of level $-l$. Following [W2, TW], we introduce the following operator

$$D(t) = \sum_{k \in \mathbb{N}} (t^{k-\frac{1}{2}} - t^{\frac{1}{2}-k})(E_{k,k} - E_{1-k,1-k}) + \frac{2}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} C.$$

Regarding d_∞ as a subalgebra of a_∞ , we have $D(t) = A(t) - A(t^{-1})$. According to [W1, Section 5.1] we have an action of d_∞ on \mathfrak{F}^{-l} , from which we obtain

$$D(t) = \sum_{p=1}^l \sum_{r \in \frac{1}{2} + \mathbb{Z}} t^r (-\gamma_{-r}^{+,p} \gamma_r^{-,p} + \gamma_{-r}^{-,p} \gamma_r^{+,p}).$$

There is also an action of the Lie group $Sp(2l)$ on \mathfrak{F}^{-l} that commutes with the action of d_∞ .

Associated to a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ of length $\leq l$, we define a highest weight $\Lambda(\lambda)$ for d_∞ to be

$$\Lambda(\lambda) = (-2l - \lambda_1 - \lambda_2) \Lambda_0^d + \sum_{k=1}^l (\lambda_k - \lambda_{k+1}) \Lambda_k^d,$$

with the convention that $\lambda_{l+1} = 0$.

Proposition 4.1. [W1, Theorem 5.2] *We have the following $(d_\infty, Sp(2l))$ -module decomposition:*

$$\mathfrak{F}^{-l} \cong \bigoplus_{\lambda} V_{\lambda}(Sp(2l)) \otimes L(d_\infty; \Lambda(\lambda), -l),$$

where $V_{\lambda}(Sp(2l))$ denotes the irreducible $Sp(2l)$ -module of highest weight λ .

The n -point d_∞ function of level $-l$ (associated to a partition λ of length $\leq l$) is defined to be

$$\mathfrak{D}_{-l}^{\lambda}(q; t_1, \dots, t_n) := \text{tr}_{L(d_\infty; \Lambda(\lambda), -l)} q^{L_0} D(t_1) \cdots D(t_n),$$

where L_0 is as usual a degree operator.

Proposition 4.2. *The function $\mathfrak{D}_{-1}^{(m)}(q; t_1, \dots, t_n)$ is given by*

$$\begin{aligned} [z^m] \sum_{\vec{\epsilon}_a \in \{\pm 1\}^n} [\vec{\epsilon}_a] \text{tr}_{\mathfrak{F}^{-1}} z^{\epsilon_{11}} q^{L_0} A(t_1^{\epsilon_1}) \cdots A(t_n^{\epsilon_n}) \\ - [z^{m+2}] \sum_{\vec{\epsilon}_a \in \{\pm 1\}^n} [\vec{\epsilon}_a] \text{tr}_{\mathfrak{F}^{-1}} z^{\epsilon_{11}} q^{L_0} A(t_1^{\epsilon_1}) \cdots A(t_n^{\epsilon_n}), \end{aligned}$$

where as before we denote $\vec{\epsilon} = (\epsilon_1, \dots, \epsilon_n)$ and $[\vec{\epsilon}] = \epsilon_1 \epsilon_2 \cdots \epsilon_n$.

Proof. Recalling that $D(t) = A(t) - A(t^{-1})$ when acting on \mathfrak{F}^{-1} , we have

$$\begin{aligned}
& \text{tr}_{\mathfrak{F}^{-1}} z^{e_{11}} q^{L_0} D(t_1) \cdots D(t_n) \\
&= \prod_{j=1}^n \text{tr}_{\mathfrak{F}^{-1}} z^{e_{11}} q^{L_0} (A(t_j) - A(t_j^{-1})) \\
&= \sum_{\vec{\epsilon} \in \{\pm 1\}^n} [\vec{\epsilon}] \text{tr}_{\mathfrak{F}^{-1}} z^{e_{11}} q^{L_0} A(t_1^{\epsilon_1}) A(t_2^{\epsilon_2}) \cdots A(t_n^{\epsilon_n}) \\
&= \sum_{m \in \mathbb{Z}} z^m \sum_{\vec{\epsilon} \in \{\pm 1\}^n} [\vec{\epsilon}] \text{tr}_{\mathfrak{F}_{(m)}^{-1}} q^{L_0} A(t_1^{\epsilon_1}) A(t_2^{\epsilon_2}) \cdots A(t_n^{\epsilon_n}).
\end{aligned}$$

Applying the trace of $z^{e_{11}} q^{L_0} D(t_1) \cdots D(t_n)$ to both sides of Proposition 4.1 when $l = 1$ yields

$$\text{tr}_{\mathfrak{F}^{-1}} z^{e_{11}} q^{L_0} D(t_1) \cdots D(t_n) = \sum_{m \in \mathbb{Z}_+} \text{ch}_{(m)}^{sp}(z) \cdot \text{tr}_{L(d_\infty; \Lambda(m), -1)} q^{L_0} D(t_1) \cdots D(t_n).$$

Combining the above equations, substituting the character formula $\text{ch}_{(m)}^{sp}(z) = (z^{m+1} - z^{-(m+1)})/(z - z^{-1})$, and clearing denominators give us

$$\begin{aligned}
& (z - z^{-1}) \sum_{m \in \mathbb{Z}} z^m \sum_{\vec{\epsilon} \in \{\pm 1\}^n} [\vec{\epsilon}] \text{tr}_{\mathfrak{F}^{-1}} q^{L_0} A(t_1^{\epsilon_1}) A(t_2^{\epsilon_2}) \cdots A(t_n^{\epsilon_n}) \\
&= \sum_{k \in \mathbb{Z}_{\geq 0}} (z^{k+1} - z^{-(k+1)}) \cdot \text{tr}_{L(d_\infty; \Lambda(k), -1)} q^{L_0} D(t_1) \cdots D(t_n).
\end{aligned}$$

Now the proposition follows. \square

Theorem 4.1. *The function $\mathfrak{D}_{-l}^\lambda(q; t_1, \dots, t_n)$ is given by*

$$\sum_{\sigma \in W(C_l)} (-1)^{\ell(\sigma)} \mathfrak{D}_{-1}^{(k_1)}(q; t_1, \dots, t_n) \cdots \mathfrak{D}_{-1}^{(k_l)}(q; t_1, \dots, t_n),$$

where $k_i = (\lambda + \rho_C - \sigma(\rho_C), \varepsilon_i)$.

Proof. Recall the $Sp(2l)$ -character formula (cf. [FH, 24.18])

$$\text{ch}_\lambda^{sp}(z_1, \dots, z_l) = \frac{\left| z_j^{\lambda_i + l - i + 1} - z_j^{-(\lambda_i + l - i + 1)} \right|}{\left| z_j^{l - i + 1} - z_j^{-(l - i + 1)} \right|}.$$

The proof proceeds in the same way as that of Theorem 2.5, using now Proposition 4.1, replacing ch_λ^{gl} with ch_λ^{sp} , clearing denominators and comparing coefficients. Note that S_l therein is replaced with the Weyl group $W(C_l)$. \square

4.3. The q -dimension of a d_∞ -module of level $-l$. Denote by ${}^d\mathbf{Q}_{-l}^\lambda(q)$ the q -dimension of $L(d_\infty; \Lambda(\lambda); -l)$.

Proposition 4.3. *Let $k \in \mathbb{Z}_+$. The q -dimension of the irreducible d_∞ -module of highest weight $\Lambda(k)$ and level -1 is*

$${}^d\mathbf{Q}_{-1}^{(k)}(q) = \frac{1}{(q)_\infty^2} \sum_{m \geq 0} (-1)^m q^{\frac{1}{2}m(m+1)} \left(q^{k(m+\frac{1}{2})} - q^{(k+2)(m+\frac{1}{2})} \right).$$

Proof. This proof is similar to that of Proposition 2.3. By Proposition 4.1 and the $Sp(2)$ -character formula (and clearing the Weyl denominator $(z - z^{-1})$), we arrive at the following identity

$$\sum_{r \in \mathbb{Z}_+} (z^{r+1} - z^{-(r+1)}) {}^d\mathbf{Q}_{-1}^{(r)}(q) = (z - z^{-1}) \frac{1}{(q)_\infty^2} \sum_{r \in \mathbb{Z}} \sum_{m=0}^{\infty} z^r (-1)^m q^{\frac{1}{2}m(m+1)} q^{|r|(m+\frac{1}{2})}.$$

The result now follows by comparing the coefficient of z^{k+1} on both sides. \square

Theorem 4.2. *The q -dimension of the irreducible d_∞ -module of highest weight $\Lambda(\lambda)$ and level $-l$ is*

$${}^d\mathbf{Q}_{-l}^\lambda(q) = \sum_{\sigma \in W(C_l)} (-1)^{\ell(\sigma)} {}^d\mathbf{Q}_{-1}^{(k_1)}(q) \cdots {}^d\mathbf{Q}_{-1}^{(k_l)}(q),$$

where $k_i = (\lambda + \rho_C - \sigma(\rho_C), \varepsilon_i)$.

Proof. This follows from an appropriate change of the Weyl groups from S_l to $W(C_l)$ in the proof of Theorem 2.7. \square

4.4. The n -point d_∞ -functions of level $-l + \frac{1}{2}$. Introduce a neutral fermionic field $\phi(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \phi_n z^{-n-\frac{1}{2}}$ which satisfies the following commutation relations:

$$[\phi_m, \phi_n]_+ = \delta_{m,-n}, \quad m, n \in \mathbb{Z} + \frac{1}{2}.$$

Denote by $\mathfrak{F}^{\frac{1}{2}}$ the Fock space of $\phi(z)$. According to [W1, Section 6.1], the Fock space $\mathfrak{F}^{-l+\frac{1}{2}} = \mathfrak{F}^{-l} \otimes \mathfrak{F}^{\frac{1}{2}}$ admits the commuting actions of $\mathfrak{osp}(1, 2l)$ and d_∞ .

Associated to a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ of length $\leq l$, we define a highest weight $\Lambda(\lambda)$ for d_∞ :

$$\Lambda(\lambda) = (-2l + 1 - \lambda_1 - \lambda_2) \Lambda_0^d + \sum_{k=1}^l (\lambda_k - \lambda_{k+1}) \Lambda_k^d,$$

where λ is a partition and we again take the convention that $\lambda_{l+1} = 0$.

Proposition 4.4. [W1, Theorem 6.1] *We have the following $(\mathfrak{osp}(1, 2l), d_\infty)$ -module decomposition*

$$\mathfrak{F}^{-l+\frac{1}{2}} \cong \bigoplus_{\lambda} V_{\lambda}(\mathfrak{osp}(1, 2l)) \otimes L(d_\infty; \Lambda(\lambda), -l + \frac{1}{2}),$$

where the summation is over all partitions λ with $\ell(\lambda) \leq l$.

We can express the operator $D(t)$ acting on $\mathfrak{F}^{-l+\frac{1}{2}}$ as

$$D(t) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} \sum_{p=1}^l t^r (-\gamma_{-r}^{+,p} \gamma_r^{-,p} + \gamma_{-r}^{-,p} \gamma_r^{+,p}) + \sum_{r \in \frac{1}{2} + \mathbb{Z}} t^r \phi_{-r} \phi_r.$$

Recall that the n -point d_∞ -function of level $\frac{1}{2}$, $\mathfrak{D}_{\frac{1}{2}}^{(0)}(q; t_1, \dots, t_n)$, has been computed in [TW, W2].

Theorem 4.3. *The n -point d_∞ -function of level $-l + \frac{1}{2}$, $\mathfrak{D}_{-l+\frac{1}{2}}^{\lambda}(q; t_1, \dots, t_n)$, is given by*

$$\mathfrak{D}_{\frac{1}{2}}^{(0)}(q; t_1, \dots, t_n) \sum_{\sigma \in W(B_l)} (-1)^{\ell(\sigma)} \mathfrak{D}_{-1}^{(k_1)}(q; t_1, \dots, t_n) \cdots \mathfrak{D}_{-1}^{(k_l)}(q; t_1, \dots, t_n),$$

where $k_i = (\lambda + \rho_B - \sigma(\rho_B), \epsilon_i)$.

Proof. The proof is similar to that of Theorem 2.5, where we replace ch_{λ}^{gl} with $\text{ch}_{\lambda}^{osp}$ (using Lemma 3.2). Note that $z_i^{e_{ii}}$ does not act on $\mathfrak{F}^{\frac{1}{2}}$ which produces the appearance of the factor $\mathfrak{D}_{\frac{1}{2}}^{(0)}(q; t_1, \dots, t_n)$ in the result. \square

4.5. The q -dimension of a d_∞ -module of level $-l + \frac{1}{2}$.

Theorem 4.4. *The q -dimension of the irreducible d_∞ -module of highest weight $\Lambda(\lambda)$ and level $-l + \frac{1}{2}$ is*

$$dQ_{-l+\frac{1}{2}}^{\lambda}(q) = (-q^{\frac{1}{2}})_{\infty} \sum_{\sigma \in W(B_l)} (-1)^{\ell(\sigma)} dQ_{-1}^{(k_1)}(q) \cdots dQ_{-1}^{(k_l)}(q),$$

where $k_i = (\lambda + \rho_B - \sigma(\rho_B), \epsilon_i)$.

Proof. The proof proceeds as that of Theorem 2.7 with a few changes. First, we substitute the Weyl group of type B for S_l . Now note that $z_i^{e_{ii}}$ does not act on $\mathfrak{F}^{\frac{1}{2}}$, which produces a factor of $\text{tr}_{\mathfrak{F}^{\frac{1}{2}}} q^{L_0}$ out front, which is equal to $(-q^{\frac{1}{2}})_{\infty}$. \square

REFERENCES

- [BO] S. Bloch and A. Okounkov, *The characters of the infinite wedge representation*, Adv. Math. **149** (2000), 1–60.
- [BCMN] P. Bouwknegt, A. Ceresole, J. McCarthy and P. van Nieuwenhuizen, *Extended Sugawara construction for the superalgebras $SU(M+1|N+1)$. I. Free-field representation and bosonization of super Kac-Moody currents*, Phys. Rev. D **39** (1989), 2971–2986.
- [BtD] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Springer, 1985.
- [CW] S.-J. Cheng and W. Wang, *The Bloch-Okounkov correlation functions at higher levels*, Transform. Groups **9** (2004), 133–142.
- [DJKM1] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *Operator approach to the Kadomtsev-Petviashvili equation. Transformation groups for soliton equations III*, J. Phys. Soc. Japan **50** (1981), 3806–3812.
- [DJKM2] ———, *A new hierarchy of soliton equations of KP-type. Transformation groups for soliton equations IV*, Physics **4D** (1982), 343–365.
- [FeF] B. Feigin and E. Frenkel, *Semi-infinite Weil complex and the Virasoro algebra*, Comm. Math. Phys. **137** (1991), 617–639.
- [FF] A. Feingold and I. Frenkel, *Classical affine algebras*, Adv. Math. **56** (1985), 117–172.
- [FH] W. Fulton and J. Harris, *Representation Theory: A First Course*, Springer, 1991.
- [GR] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics **96**, Cambridge University Press, 2004.
- [Ho1] R. Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989), 539–570.
- [Ho2] ———, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, Schur Lect. (Tel Aviv)(1992), 1–182, Israel Math. Conf. Proc. **8**.
- [K1] V. Kac, *Vertex algebras for beginners*, second edition, University Lecture Series **10**, AMS, Providence, RI, 1998.
- [K2] ———, *Representations of classical Lie Superalgebras*, Lecture Notes in Math. **676**, pp. 597–626, Springer, 1978.
- [KR] V. Kac and A. Radul, *Representation theory of the vertex algebra $W_{1+\infty}$* , Transform. Groups **1** (1996), 41–70.
- [KWY] V. Kac, C.H. Yan, and W. Wang, *Quasifinite representations of classical Lie subalgebras of $W_{1+\infty}$* , Adv. Math. **139** (1998), 56–140.
- [Mil] A. Milas, *Formal differential operators, vertex operator algebras and zeta-values, II*, J. Pure Appl. Algebra **183** (2003), 191–244.
- [Ok] A. Okounkov, *Infinite wedge and random partitions*, Selecta Math., New Series **7** (2001), 1–25.
- [TW] D. Taylor and W. Wang, *The Bloch-Okounkov correlation functions of classical type*, Commun. Math. Phys. (to appear), math.RT/0609036.
- [W1] W. Wang, *Duality in infinite dimensional Fock representations*, Commun. Contemp. Math. **1** (1999), 155–199.
- [W2] ———, *Correlation functions of strict partitions and twisted Fock spaces*, Transform. Groups **9** (2004), 89–101.

INSTITUTE OF MATHEMATICS, ACADEMIA SINICA, TAIPEI, TAIWAN 11529

E-mail address: `chengsj@math.sinica.edu.tw`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VA 22904;
(*New address after August 2007*) DEPARTMENT OF MATHEMATICS, COMPUTER SCIENCE, AND
PHYSICS, ROANOKE COLLEGE, SALEM, VA 24153

E-mail address: `taylor@roanoke.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VA 22904

E-mail address: `ww9c@virginia.edu`